ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN-POLONIA

VOL. XLVII, 13 SECTIO A 1993

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A Survey on Uniformly Convex and Uniformly Starlike Functions

1. Introduction. The present article is a survey trying to cover the most important results hitherto known about the functions called uniformly convex and uniformly starlike. These classes of functions were introduced by A.W. Goodman in [5] and followed up in [4]. Most of the results in [4] and [5] can also be found in the survey article [6] on open problems. Later, other authors have taken interest in these classes, and several papers have been written. The results in this paper can, with few exceptions, be found elsewhere, and therefore their proofs are omitted, but an extensive list of references is included. Before proceeding we give some notations and definitions.

Let $\mathcal{A}_0$ be the class of functions analytic in the unit disk U normalized by $f(0) = f'(0) - 1 = 0$, and let, as usual, $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}_0$. Let S^*_α $(0 \leq \alpha \leq 1)$ denote the class of functions in $\mathcal{A}_0$ that are starlike of order α, and let $\mathcal{K}$ denote the class of convex functions. Then we have the classical analytic characterizations.

$$f \in S^*_\alpha \iff \operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha, \quad z \in U$$

and

$$f \in \mathcal{K} \iff \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0, \quad z \in U.$$

In addition to these well known classes we introduce three classes of functions that will be our main interest in this paper.

Definition . Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}_0$.

(a) $f \in UST$ if and only if

$$\operatorname{Re} \frac{f(z) - f(\zeta)}{(z-\zeta)f'(z)} \geq 0, \quad (z,\zeta) \in U \times U. \tag{1.1}$$

(b) $f \in UCV$ if and only if

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad (z,\zeta) \in U \times U. \tag{1.2}$$

(c) $f \in S_p$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)}, \quad z \in U. \tag{1.3}$$

The classes UST and UCV have a natural geometric interpretation in the sense that $f \in UST$ if and only if the image of every circular arc in U with center ζ also in U, is starlike with respect to $f(\zeta)$, and $f \in UCV$ if and only if the image of such an arc is convex. The connection between the geometric and analytic characterization is easily established by the standard methods where convexity and starlikeness are interpreted as monotonic rotation of the tangent vector and the radius vector respectively. These are the classes of *uniformly starlike* and *uniformly convex* functions as introduced by Goodman. Note that if we take $\zeta = 0$ in (1.1) and (1.2) we have the usual classes of starlike and convex functions, and that if we let $\zeta \to z$ the conditions are trivially fulfilled. Goodman's definition of UST and UCV was motivated by the following question from B. Pinchuk: Will a function $f \in \mathcal{S}_0^*$ map a circle contained in U, with center $\zeta \neq 0$, to a curve that is starlike with respect to $f(\zeta)$? Simple examples in [5] show that the answer to this question is no. The same problem was independently studied by J.E. Brown [3]. His work goes much deeper into this particular question than Goodman's work does, but does not lead to the definition of uniform starlikeness. For a given $|\zeta| = r < 1$ Brown determines a value $\rho = \rho(r)$ such that every $f \in \mathcal{S}$ maps each disk $|z - \zeta| < \rho$ contained in U onto a region which is starlike with respect to $f(\zeta)$. The value of ρ that he obtains is sharp for the whole class $\mathcal{S}$ and is, he remarks, the best known estimate for $\mathcal{S}_0^*$. To the best of our knowledge, a sharp bound for ρ in the class $\mathcal{S}_0^*$ is still not known. The value of ρ as a function of r cannot be given explicitely, but for any given r it can be obtained by solving a given equation. For details, we refer to [3]. Note that in the class $\mathcal{K}$ the situation is different in the sense that convex functions map circles inside of U to convex curves, and this has been known for a long time [13,25]. So the answer to the Pinchuk question with $\mathcal{S}_0^*$ replaced by $\mathcal{K}$ is yes. However, in the definitions of UST and UCV it is not a restriction that z and ζ are such that the disk $|z - \zeta| < \rho$ is contained in U. In this situation it is clear that (1.2) will not hold in general for convex functions. For a simple example take $f(z) = z/(1 - z)$ which maps every circular arc through $z = 1$ to a straight line. Now, by mapping the circular arc $\gamma(\rho) : |z - \zeta| = \rho$ for various values of ρ it is clear that if we pass a value of ρ for which $\gamma(\rho)$ goes through $z = 1$ the real part in (1.2) will change sign. Therefore, UST and UCV represent proper subclasses of $\mathcal{S}_0^*$ and $\mathcal{K}$ respectively.

One of the difficulties with UST and UCV is that their characterizations are given in terms of two variables. Ma and Minda [9] and the author [18] independently discovered that in UCV it is possible to give a characterization in terms of only one variable.

Theorem 1.1. *$f \in UCV$ if and only if*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}, \quad z \in U. \tag{1.4}$$

The proof is an application of the Minimum Principle for harmonic functions. Comparing (1.3) and (1.4) we see that we have

Corollary 1.2.

$$f \in UCV \iff zf' \in S_p. \tag{1.5}$$

Hence, the relation between UCV and S_p is given in terms of an Alexander type theorem, and, with the geometric similarities between UCV and UST in mind, it is natural to ask if there is any relation between S_p and UST. Examples given in [4,16,18] show that $UST \not\subset S_p$ and also $S_p \not\subset UST$. The relation (1.5) was the motivation for introducing the class S_p in [18].

2. Related classes of starlike functions. The class S_p fits nicely in among other special classes of starlike functions that have been investigated over the years. We see from (1.3) that the domain of values of $zf'(z)/f(z)$ for $f \in S_p$, $z \in U$, is contained in a parabola symmetric about $[\frac{1}{2}, \infty)$. Hence, we have the sharp inclusion $S_p \subset S^*_{1/2}$. Other classes of starlike functions that are natural to compare with are the so-called strongly starlike functions introduced independently by Brannan and Kirwan [2] and Stankiewicz [22,24]. A function $f \in \mathcal{A}_0$ is said to be strongly starlike of order α $(0 < \alpha \leq 1)$, here denoted STS_α, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \alpha\frac{\pi}{2}, \quad z \in U.$$

We see that we also have the sharp inclusion $S_p \subset STS_{1/2}$.

In [8] Ma and Minda gave a geometric characterization of the functions in STS_α. Before presenting their result we need some notation and definitions. For $0 < k \leq 2$ let E_k be the intersection of the two closed disks of radii $1/k$ both of which have 0 and 1 on their boundaries. For $k = 0$, E_0 is taken to be the line segment $[0, 1]$. A region Ω $(0 \in \Omega)$ is said to be k-starlike (with respect to the origin) if for every $\omega \in \Omega$, $\omega E_k \subset \Omega$. This means that ordinary starlikeness corresponds to $k = 0$, and that for $k > 0$ the region contains a certain lens-shaped region. With this notation the result from [8] can be expressed as follows.

Theorem 2.1. *Let $f \in STS_\alpha$. Then the region $f(U)$ is k-starlike with*

$$k = 2\cos\frac{\pi\alpha}{2},$$

and this value is sharp.

The maximal value of k corresponds directly to the maximal value of $|\arg zf'(z)/f(z)|$, so because of the sharp inclusion $S_p \subset STS_{1/2}$, there is a function in S_p which is at most $\sqrt{2}$-starlike. To obtain better information about the geometry of S_p one should perhaps introduce some other concept, similar in nature to k- starlikeness. It is not, however, clear what this should be.

In all these classes many properties can be derived by investigating the Carathéodory function $P(z)$ mapping U onto the maximal range of values of $zf'(z)/f(z)$. We take P to be normalized such that $P(0) = 1$ and $P'(0) > 0$. In the case of S^*_α this P is

$$P(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$$

and in the case of STS_α it is

$$P(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}.$$

We could also mention that a generalization of these classes was done by Wesołowski [27] who investigated starlike functions where the corresponding Carathéodory function is

$$P(z)=\left[\frac{1+(1-2\alpha)z}{1-z}\right]^{\beta}.$$

We observe that for $\beta=\frac{1}{2}$, $\alpha\neq 0$ the boundary of $P(U)$ is a hyperbola. In the case of S_p the corresponding Carathéodory function mapping U onto the parabolic region is seen to be

$$P(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2. \tag{2.1}$$

We include here an observation that shows how the coefficients of $P(z)$ in (2.1) can be expressed nicely in terms of special functions.

Theorem 2.2. *Let*

$$P(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2=1+\frac{2}{\pi^2}\sum_{k=1}^{\infty}A_k z^k.$$

Then

$$A_k=\frac{2}{k}\left(\Psi\left(\frac{1}{2}+k\right)+\gamma+2\log 2\right)$$

where $\Psi(x)=\Gamma'(x)/\Gamma(x)$ *and* γ *is Euler's constant.*

Proof. Using the expansion $\log\frac{1+\sqrt{z}}{1-\sqrt{z}}=2\sqrt{z}\sum_{k=0}^{\infty}z^k/(2k+1)$ we find after squaring that

$$A_k=4\sum_{m=1}^{k}\frac{1}{(2m-1)(2k-2m+1)}$$

which by a partial fraction decomposition can be written

$$A_k=\frac{2}{k}\sum_{m=1}^{k}\left(\frac{1}{2m-1}+\frac{1}{2(k-m)+1}\right).$$

We see that $\sum_{m=1}^{k}\frac{1}{2m-1}=\sum_{m=1}^{k}\frac{1}{2(k-m)+1}$, so we have

$$A_k=\frac{4}{k}\sum_{m=1}^{k}\frac{1}{2m-1}.$$

This sum is related to the Ψ-function in the following way [1, p.258]:

$$\Psi\left(\frac{1}{2}+k\right) = -\gamma - 2\log 2 + 2\sum_{m=1}^{k} \frac{1}{2m-1},$$

and the result follows.

All the classes of starlike functions that we have mentioned (except UST) can be characterized by the range of the functional $zf'(z)/f(z)$. Let Ω be a convex domain in the right half plane, symmetric about the real axis, and assume that $1 \in \Omega$. Define $S^*(\Omega)$ as the set of functions in $\mathcal{A}_0$ with the property that $zf'(z)/f(z) \in \Omega$, and let P_Ω be the (normalized) Carathéodory function mapping the unit disk onto Ω. Further we define $k_\Omega \in S^*(\Omega)$ by $zk'_\Omega(z)/k_\Omega(z) = P_\Omega(z)$. All the classes described above fit into this pattern, and the function k_Ω is extremal for its class in many respects. Note that in the case S_0^* k_Ω is simply the Koebe function. With the setting just presented we have the following general result (Ma and Minda [7]).

Theorem 2.3. *For $f \in S^*(\Omega)$ we have*

$$\frac{f(z)}{z} \prec \frac{k_\Omega(z)}{z}$$

and

$$-k_\Omega(-r) \le |f(z)| \le k_\Omega(r), \quad |z| \le r < 1. \tag{2.2}$$

If $\lim_{r\to 1-} k_\Omega(r) = M < \infty$ then the functions in $S^*(\Omega)$ are bounded, and we have

$$|f(z)| \le M, \quad z \in U. \tag{2.3}$$

The Koebe constant of $S^*(\Omega)$ is given by $-k_\Omega(-1)$. Also we have

$$k'_\Omega(-r) \le |f'(z)| \le k'_\Omega(r), \quad |z| \le r < 1 \tag{2.4}$$

and

$$\left|\arg\frac{f(z)}{z}\right| \le \max_{\theta\in[0,2\pi)} \arg\left\{\frac{k_\Omega(re^{i\theta})}{re^{i\theta}}\right\}, \quad |z| \le r < 1. \tag{2.5}$$

Equality in (2.2), (2.4), (2.5) holds for some $z \ne 0$ if and only if f is a rotation of k_Ω.

Theorems of this type have previously been proved separately for each of the classes we have mentioned. In each case the technique applied is essentially the same. (See e.g. [2,9,15,18,24,28].) This technique can be unified to prove the more general theorem above as Ma and Minda have done in [7]. In fact they obtain all results, except (2.4), by assuming only that Ω is symmetric about the real axis and starlike with respect to the point $\omega = 1$. They also give some other general properties of functions from $S^*(\Omega)$ as well as of classes of convex functions obtained similarly by replacing the functional $zf'(z)/f(z)$ with $1 + zf''(z)/f'(z)$.

In some cases one has been able to compute the number M in (2.3) explicitely. In the case of the strongly starlike functions STS_α Brannan and Kirwan [2] showed that

$$M = M(\alpha) = \frac{1}{4}\exp\left\{-\Psi\left(\frac{1-\alpha}{2}\right) - \gamma\right\}$$

where again $\Psi(x)$ is the logarithmic derivative of the Gamma function and γ is Euler's constant. In [18] the author similarly found the upper bound in S_p to be

$$M = \exp\left\{\frac{14}{\pi^2}\zeta(3)\right\} \tag{2.6}$$

where $\zeta(x)$ is the Riemann Zeta function. (Note that this bound is, by (1.5), the upper bound of the derivative in UCV.) The Koebe constant in S_p is in [15] computed to be

$$K = \exp\left\{-\frac{4}{\pi^2}\int_0^{\pi/2} \frac{t^2}{\sin t}\,dt\right\} = 0.53399\ldots, \tag{2.7}$$

hence slightly larger than the Koebe constant in $\mathcal{K}$ which is $1/2$. We could mention that in [15] the class S_p was generalized by introducing a parameter α in the following way. Let $S_p(\alpha)$ be defined by

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\frac{zf'(z)}{f(z)} - \alpha. \tag{2.8}$$

Then we get a domain Ω whose boundary is a parabola with vertex $\omega = (1+\alpha)/2$. We see that for all $\alpha \in [-1, 1)$ we have $S_p(\alpha) \subset S_0^*$. If we introduce a corresponding class $UCV(\alpha)$ (uniformly convex of order α) by $g \in UCV(\alpha) \iff zg' \in S_p(\alpha)$ we observe that $UCV(\alpha) \subset \mathcal{K}$ for $\alpha \geq -1$. Now, bearing in mind the expression (1.2), this is the same as to say that

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{g''(z)}{g'(z)}\right\} \geq -1, \quad (z,\zeta) \in U \times U$$

implies

$$\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \geq 0, \quad z \in U.$$

For the generalized classes $S_p(\alpha)$ it is easy to see [15] that the bounds on the modulus are $M(\alpha) = M^{1-\alpha}$ and $K(\alpha) = K^{1-\alpha}$ where M and K are as in (2.6) and (2.7).

3. Some properties of uniformly convex functions. The relation between the classes S_p and UCV implies that results about bounds for the coefficients in one of the two classes can be directly translated to the other class. We choose here to give coefficient results in terms of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in UCV$. The first general bound on $|a_k|$ was given by Goodman [4] who showed that

$$|a_k| \leq \frac{1}{k},$$

and a better bound was given in [18], namely

$$|a_2| \leq \frac{4}{\pi^2}$$

$$|a_k| \leq \frac{8}{k(k-1)\pi^2} \prod_{j=3}^{k} \left(1 + \frac{8}{(j-2)\pi^2}\right), \quad k \geq 3,$$

where only the bound on $|a_2|$ is sharp. In [9] we find the bound $|a_k| \leq (\frac{8}{3\pi^2} + \frac{128}{3\pi^4})/k$, also not sharp. The best results on coefficients in UCV have been obtained by Ma and Minda [9,10] who gave sharp bounds for the first six coefficients and a sharp order of growth for $|a_k|$. Let $F \in UCV$ be the function for which

$$1 + \frac{zF''(z)}{F(z)} = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$$

where

$$F(z) = z + \sum_{k=1}^{\infty} A_k z^k,$$

and let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in UCV$. Then we have the following theorem [9,10].

Theorem 3.1.

(a)

$$|a_k| \leq A_k, \quad k = 1, 2, 3, 4, 5, 6.$$

Equality holds if and only if f is a rotation of F.

(b) *The sharp order of growth for $|a_k|$ is*

$$|a_k| = \mathcal{O}(1/k^2).$$

The numbers A_k can be computed, and the exact values can be found in [10]. This theorem has an interesting consequence because for the function F one can show [9] that $A_k = \mathcal{O}((\log k)^2/k^3)$ for large k which, in comparison with statement (b) of Theorem 3.1, means that there exists a function $f \in UCV$ such that for k large enough, $|a_k| > A_k$. It is not known how large k must be for this to happen. Also, no sharp general bounds on $|a_k|$ in UCV are known. The order of growth statement in Theorem 3.1 (b) can in fact be obtained from a more general statement. In [14] the class S_p was generalized in a different manner than (2.8), introducing

$$\mathcal{P}(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - \alpha \right| \le \operatorname{Re} \frac{zf'(z)}{f(z)} + \alpha, \quad z \in U, \quad 0 < \alpha < \infty \right\}.$$

Now the domain Ω is bounded by a parabola with vertex at the origin and focus at some point on the positive real axis, so clearly $S_p(\alpha) \subset \mathcal{P}(\beta)$ for some number β. One of the results obtained in [14] is

Theorem 3.2. *Let $f \in \mathcal{P}(\alpha)$. Then for some complex valued Borel measure μ on $|x| = 1$, f can be written as*

$$f(z) = \int_{|x|=1} \log \frac{1}{1 - xz} d\mu(x).$$

As a consequence of this we see that if $f(z) = \sum_{k=1}^{\infty} a_k z^k$ we have

$$a_n = \frac{1}{n} \int_{|x|=1} x^n d\mu(x),$$

and hence that the order of growth of $|a_n|$ is $\mathcal{O}(1/n)$. The functions in $\mathcal{P}(\alpha)$ are starlike functions, so therefore the growth of the coefficients of the corresponding convex functions is $\mathcal{O}(1/n^2)$.

Ma and Minda also investigated the functional $|\mu a_2^2 - a_3|$ and obtained upper bounds for various values of μ. We refer to [10] for the complete results, but include here the special case $\mu = 1$ in comparison with the corresponding result for the class $\mathcal{K}$ which is due to Trimble [26].

Theorem 3.3. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.*
(a) If $f \in UCV$ then $|a_2^2 - a_3| + \frac{12+\pi^2}{36}|a_2|^2 \le \frac{4}{3\pi^2}$.
(b) If $f \in \mathcal{K}$ then $|a_2^2 - a_3| + \frac{1}{3}|a_2|^2 \le \frac{1}{3}$.

To finish this section we mention a convolution result which was proved in [15].

Theorem 3.4. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S_{1/2}^*$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_p$ then $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \in S_p$.*

Compare this with the famous result of Ruscheweyh and Sheil-Small [20] (Pólya-Schoenberg conjecture): $f \in \mathcal{K}$, $g \in S_0^* \Rightarrow f * g \in S_0^*$. One way of expressing this is that $\mathcal{K}$ is a multiplier family from S_0^* to S_0^*, and that the (much larger) class $S_{1/2}^*$ is a multiplier family from S_p to S_p. The basic explanation of these convolution properties is contained in the following lemma (See [19, pp.54–55]).

Lemma 3.5. *Let $\mathcal{R}_\alpha$ be the class of functions in A_0 given by*

$$f \in \mathcal{R}_\alpha \iff f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*_\alpha. \tag{3.1}$$

Then for $f \in \mathcal{R}_\alpha, g \in S^*_\alpha$ and any function H analytic in U we have

$$\frac{f * gH}{f * g}(U) \subset \overline{\text{co}}\, H(U).$$

In proving Theorem 3.4 this result was applied in the case $\alpha = 1/2$. Turning now to the more general classes $S^*(\Omega_\alpha)$ (as in Theorem 2.3) where Ω_α is assumed to be convex and all $\omega \in \Omega_\alpha$ have $\operatorname{Re} \omega \geq \alpha$ we can prove

Theorem 3.6. *If $f \in \mathcal{R}_\alpha$ and $g \in S^*(\Omega_\alpha)$ then $f * g \in S^*(\Omega_\alpha)$.*

Proof. Let $zg'(z)/g(z)$ play the role of $H(z)$ in Lemma 3.5 and we get

$$\frac{z(f*g)'}{f*g}(U) = \frac{f*g\frac{zg'}{g}}{f*g}(U) \subset \frac{zg'}{g}(U) \subset \Omega_\alpha,$$

hence $f * g \in S^*(\Omega_\alpha)$.

Remark. Since $h \in UCV \iff g = zh' \in S_p$ and $z(f*h)' = f * zh'$ it is clear that in Theorem 3.4 we can substitute S_p with UCV. Theorem 3.6 was presented in a slightly more special form in [7]. Looking at the definition of $\mathcal{R}_\alpha$ in (3.1) we see that $\mathcal{R}_0 = \mathcal{K}$ and $\mathcal{R}_{1/2} = S^*_{1/2}$. The classes $\mathcal{R}_\alpha$ are known as prestarlike functions, and they play an important role in convolution theory. For further study of these functions we refer to [19].

Finally we mention a radius result for UCV which can be found in [17]. Recall that the radius of convexity in $\mathcal{S}$ is $2 - \sqrt{3} = 0.2679\ldots$

Theorem 3.7. *The radius of uniform convexity in $\mathcal{S}$ is $\frac{1}{3}(4 - \sqrt{13}) = 0.1314\ldots$*

4. Uniformly starlike functions. There are so far not so many results about uniformly starlike functions, much due to the fact that we basically do not have a simpler characterization of the class than the one given in (1.1) which in use leads to very complicated computations. (A slight simplification can be made, because an application of the Minimum Principle [16] shows that it suffices to examine z and ζ with $|z| = |\zeta|$.) Sakaguchi [21] introduced in 1959 a class which he called starlike with respect to symmetrical points (SSP), which can be defined as

$$SSP = \left\{ f \in \mathcal{S} : \operatorname{Re} \frac{f(z) - f(-z)}{zf'(z)} \geq 0, \quad z \in U \right\}.$$

We see that this class is related to UST in the sense that we get the SSP condition if we take $\zeta = -z$ in (1.1), so clearly $UST \subset SSP$. The class SSP contains all convex functions [21], but UST does not. It was proved in [5] that the function $z/(1 - Az)$ is in UST if and only if $|A| \leq 1/\sqrt{2}$. In fact the number $1/\sqrt{2}$ is the radius of uniform starlikeness for the whole class $\mathcal{K}$ as the following theorem [12, 16] shows.

Theorem 4.1. *Let $f \in \mathcal{K}$. Then we have*
(a) $\frac{1}{r}f(rz) \in UST$ *if* $r \leq \frac{1}{\sqrt{2}}$.
(b) $\frac{1}{r}f(rz) \in S_p$ *if* $r \leq \frac{1}{\sqrt{2}}$.
In both cases the number $1/\sqrt{2}$ is sharp.

It is interesting to note that these two radii coincide in the class $\mathcal{K}$ despite the fact that there is no inclusion between the classes UST and S_p. It is also an interesting observation that the radius of (normal) starlikeness in the closed convex hull of $\mathcal{K}$ is $1/\sqrt{2}$, as was proved by MacGregor [11].

Looking at radius results in other classes, we observe that UST and S_p radii differ. We have e.g. the following result.

Theorem 4.2. *Let $f \in \mathcal{S}$. Then we have*
(a) $\frac{1}{r}f(rz) \in UST$ *if* $r \leq 0.3691\ldots$.
(b) $\frac{1}{r}f(rz) \in S_p$ *if* $r \leq 0.3321\ldots$.
The given numbers are sharp.

Part (a) of this thorem is proved in [17] and part (b) is proved in [18]. In both cases the upper bounds on r are given as solutions of a certain equation or system of equations. To indicate the difficulties involved in doing computations with the UST condition (1.1), we could mention that we have not been able to compute the radius of uniform starlikeness in $\mathcal{S}_0^*$. We do not even know the exact largest value of $|z|$ for which the Koebe function is uniformly starlike. The best result so far is [17] that the UST radius r_0^* in $\mathcal{S}_0^*$ lies in the interval

$$0.369 < r_0^* < 1/\sqrt{7} < 0.378. \tag{4.1}$$

The S_p radius in $\mathcal{S}_0^*$ is, however, easily seen to be $1/3$ [18], which is equal to the $\mathcal{S}_{1/2}^*$ radius. The lower bound in (4.1) is of course the number from Theorem 4.2 (a), and the upper bound $(1/\sqrt{7})$ is the largest value of $|z|$ for which the Koebe function has the SSP property.

Let r_0 be the UST radius in $\mathcal{S}$ as given in Theorem 4.2 (a), and let $k(z)$ be the Koebe function. Then certainly $g(z) = k(r_0 z)/r_0 \in UST$, but $g \notin \mathcal{S}_{1/2}^*$, since the $\mathcal{S}_{1/2}^*$ radius in $\mathcal{S}_0^*$ is sharp for the Koebe function. Therefore $UST \not\subset \mathcal{S}_{1/2}^*$. An open problem is to find the largest $\alpha \geq 0$ such that $UST \subset \mathcal{S}_\alpha^*$.

Merkes and Salmassi [12] have a radius result for the classes $\mathcal{R}_\alpha$ that was defined in (3.1). In their paper they also discuss some properties of a subclass of UST obtained by replacing the derivative in (1.1) by a difference quotient. They show that the subclass that they define has a number of properties in common with the class UST itself. The paper also includes a result about the length of images of circles under UST mappings.

The best coefficient result for UST so far was proved by Charles Horowitz (published in [5]), and is as follows.

Theorem 4.3. *Let* $f(z)=\sum_{k=1}^{\infty} a_k z^k \in UST$. *Then*

$$|a_k| \le \frac{2}{k}, \quad k=2,3,\ldots. \tag{4.2}$$

This is considerably better than the bound for the larger class SSP which is $|a_k| \le 1$ (sharp) [21]. The bound in (4.2) was obtained by showing that if $f \in UST$ then for some real number α, $\mathrm{Re}\{e^{i\alpha}f'(z)\} \ge 0$, $z \in U$. The class of functions for which the derivative lies in a halfplane is of course much larger than UST, so the coefficient problem in UST is still unsolved.

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(received September 1, 1993)