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Remarks on Jensen's Inequality for Operator Convex Functions

Abstract. A continuous real-valued function g is said to be operator convex on an interval J if $f(sA+tB) \leq sf(A)+tf(B)$ holds for any positive s,t with $s+t=1$ and self-adjoint operators A and B with spectra contained in J. Several results which are valid for real convex functions are extended on operator convex functions.

1. Introduction. Z. Świętochowski [17] proved the following result:

Let $C_1,\dots,C_n$, be bounded positive operators. Then

$$C_1^{-1}+\cdots+C_n^{-1} \geq n^2(C_1+\cdots+C_n)^{-1} \tag{1}$$

with equality if $C_1=\dots=C_n$.

Here the inequality $A \geq B$ means that $A-B$ is a positive operator.

Note that (1) is a simple consequence of Jensen's inequality for operator convex functions. A continuous real valued function g is operator monotone on an interval J if $g(A) \leq g(B)$ for self-adjoint operators A and B such that $A \leq B$ and their spectra are contained in J. A function f is operator convex on J if

$$f(sA+tB) \leq sf(A)+tf(B) \tag{2}$$

for positive numbers s and t with $s+t=1$ and self-adjoint operators A and B with spectra contained in J. A function f is operator concave if $-f$ is operator convex on J. It is known that if f is operator monotone on $(0,\infty)$, it is also operator concave.

We denote by $S(I)$ the set of all self-adjoint operators on a Hilbert space whose spectra are contained in an interval I.

2. Jensen's and related inequalities. As in the case of classical convex functions, we can get by mathematical induction from (2), the well-known Jensen inequality:

Theorem 1. *Let $C_i \in S(I)$, $w_i > 0$, $i=1,\dots,n$ and $W_n=\sum_{i=1}^n w_i$. Then for every operator convex function f on I, we have*

$$f\Big(\frac{1}{W_n}\sum_{i=1}^n w_iC_i\Big) \leq \frac{1}{W_n}\sum_{i=1}^n w_if(C_i). \tag{3}$$

Of course we have the reverse inequality for a concave function.

Many results which are valid for real convex functions are also valid for operator convex functions with the same proofs. Here, we give such results with references to the real case.

Theorem 2. *Let w be a real n-tuple such that*

$$w_1 > 0\,, \quad w_i < 0 \quad (i = 2, \dots, n)\,, \quad W_n > 0\,. \tag{4}$$

If $C_i \in S(I)$, $i = 1, \dots, n$, $\frac{1}{W_n}\sum_{i=1}^n w_i C_i \in S(I)$, then we have the reverse inequality in (3), for every operator convex function f on I.

Now let us consider an index set function

$$F(J) = W_J f(A_J(C:w)) - \sum_{i \in J} w_i f(C_i)$$

where

$$W_J = \sum_{i \in J} w_i\,, \quad A_J(C;w) = \frac{1}{W_J}\sum_{i \in J} w_i C_i\,.$$

Theorem 3. *Let f be an operator convex function on I, J and K are two finite nonempty subsets of N such that $J \cap K = \phi$, $w = (w_i)_{i \in J \cup K}$ and $C = (C_i)_{i \in J \cup K}$ are such that $C_i \in S(I)$, $w_i \in R (i \in J \cup K)$, $W_{J \cup K} > 0$, $A_T(C;w) \in S(I)$ $(T = J, K, J \cup K)$. If $W_J > 0$ and $W_K > 0$, then*

$$F(J \cup K) \le F(J) + F(K)\,. \tag{5}$$

If $W_J W_K < 0$, we have the reverse inequality in (5).

Theorem 4. *If $w_i > 0$, $i = 1, \dots, n$, $I_k = \{1, \dots, k\}$, then*

$$F(I_n) \le F(I_{n-1}) \le \dots \le F(I_2) \le 0\,, \tag{6}$$

but if (4) is valid and $A_{I_n}(C;w) \in S(I)$ then the reverse inequalities in (6) are valid.

Theorems 2-4 in the real case are obtained in [4], [9], [16].

Theorem 5 [10]. *Let the conditions of* Theorem 1 *be fulfilled. Then*

$$\begin{aligned} f\Big(\frac{1}{W_n}\sum_{i=1}^n w_i C_i\Big) &= f_{n,n} \le \dots \le f_{k+1,n} \le f_{k,n} \le \dots \le f_{1,n} \\ &= \frac{1}{W_n}\sum_{i=1}^n w_i f(C_i)\,, \end{aligned} \tag{7}$$

where

$$f_{k,n} := \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1\le i_1<\ldots<i_k\le n} (w_{i_1}+\cdots+w_{i_k}) f\left(\frac{w_{i_1}C_{i_1}+\cdots+w_{i_k}C_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right)$$

Theorem 6. *Let the condition of* Theorem 1 *be fulfilled. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_iC_i\right) \le \ldots \le \overline{f}_{k+1,n} \le \overline{f}_{k,n} \le \ldots \le \overline{f}_{1,n} = \frac{1}{W_n}\sum_{i=1}^n w_i f(C_i) \quad , \tag{8}$$

where

$$\overline{f}_{k,n} = \frac{1}{\binom{n+k-1}{k-1} W_n} \sum_{1\le i_1<\ldots<i_k\le n} (w_{i_1}+\cdots+w_{i_k}) f\left(\frac{w_{i_1}C_{i_1}+\cdots+w_{i_k}C_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right).$$

Theorem 7 [12]. *Let the conditions of* Theorem 1 *be fulfilled. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_iC_i\right) \le \underline{f}_{k+1,n} \le \underline{f}_{k,n} \le \cdots \le \underline{f}_{1,n} = \frac{1}{W_n}\sum_{i=1}^n w_i f(C_i) \quad , \tag{9}$$

where $1 \le k \le n-1$, *and*

$$\underline{f}_{k,n} = \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1} w_{i_1}\ldots w_{i_k}) f\left(\frac{1}{k}(C_{i_1}+\cdots+C_{i_k})\right) .$$

Theorem 8 [5], [13]. *Let the condition of* Theorem 1 *be fulfilled and let* $q_i > 0$, $i = 1,\ldots,k$ *with* $Q_k := \sum_{i=1}^k q_i$. *Then*

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^n w_iC_i\right) &\le \frac{1}{W_n^k}\sum_{i_1,\ldots,i_k}^n w_{i_1}\ldots w_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^k q_jC_{i_j}\right) \\ &\le \frac{1}{W_n}\sum_{i=1}^n w_i f(C_i) \quad . \end{aligned} \tag{10}$$

Theorem 9 [6], [14]. *Let the conditions of* Theorem 1 *be fulfilled and let* $\tilde{C} = \frac{1}{W_n}\sum_{i=1}^n w_iC_i$, $t_i \in [0,1]$, $i = 1, \ldots, k-1$. *Then*

$$\begin{aligned} & f\left(\frac{1}{W_n}\sum_{i=1}^n w_iC_i\right) \le \tilde{f}_{n,1} \le \cdots \le \tilde{f}_{n,k-1} \\ & \le \frac{1}{W_n^k}\sum_{i_1,\ldots,i_k=1}^n w_{i_1}\ldots w_{i_k} f(C_{i_1}(1-t_1) + \sum_{j=1}^{k-2} C_{i_j}(1-t_{j+1})t_1\ldots t_j \\ & + C_{i_k}t_1\ldots t_{k-1} \le \frac{1}{W_n}\sum_{i=1}^n w_i f(C_i) , \end{aligned} \tag{11}$$

where

$$\bar{f}_{n,k} = \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1}^{n} w_{i_1} \ldots w_{i_k} f(C_{i_1}(1-t_1) + \sum_{j=1}^{k-1} C_{i_j}(1-t_{j+1})t_1 \ldots t_j + \bar{C} t_1 \ldots t_k).$$

Theorem 10 [7]. *Let a function g be defined by*

$$g(x) = \sum_{i=1}^{n} \frac{1}{q_i} f(q_i x A - i + (r-x) \sum_{k=1}^{n} A_k)$$

where $q_i > 0$, $i = 1, \ldots, n$, *with* $\sum_{k=1}^{n}(1/q_k) = 1$, $r \in R$, $q_i x A_i + (r-x) \sum_{i=1}^{n} A_k \in S(I)$, $i = 1, \ldots, n$ *for all* x *from an interval* J *from* R.

If $|x| \le |y|$ $(xy > 0, y \in J)$, *then*

(12) $$g(x) \le g(y) .$$

The function g *is also convex.*

Remark. Using the substitutions: $1/q_i \to w_i (\sum_{i=1}^{n} w_i = 1)$, $q_i A_i \to X_i$, $r = 1$, we get that (12) is also valid if

$$g(x) = \sum_{i=1}^{n} w_i f(x X_i + (1-x) \sum_{k=1}^{n} w_k X_k) .$$

Remark. For some further generalizations of some of the previous results, see [13] and the references given there.

3. Some inequalities for means. Note that inequality (1) is, in fact, the well-known inequality between the harmonic and arithmetic means. We can, therefore, consider generalizations to means of arbitrary orders.

We consider a power mean of stricly positive operators $C = \{C_i\}$, with weights $w = \{w_i\}$, $w_i > 0$, $i = 1, \ldots, n$ i.e.

(13) $$M_n^{[r]}(C; w) = (\frac{1}{W_n} \sum_{i=1}^{n} w_i C_i^r)^{\frac{1}{r}} .$$

If $w = (1, 1, \ldots, 1)$ we write $M_n^{[r]}(C)$.

The following results are proved in [15]:

(i) $r \ge s$, $s \notin (-1, 1)$, $r \notin (-1, 1)$ implies

$$M_n^{[r]}(C) \ge M_n^{[s]}(C) ;$$

(ii) For a finite set of positive operators

$$M_n^{[2r]}(C) \geq M_n^{[r]}(C) \quad \text{for } r \geq 1/2 .$$

Moreover, we have the following ([16]):
Let A dnote a set of strictly positive operators. Then

$$M_n^{[r]}(C;w) \geq M_n^{[s]}(C;w) \tag{14}$$

is valid if either
(a) $r \geq s$, $r \notin (-1,1)$, $s \notin (-1,1)$; or
(b) $r \geq 1 \geq s \geq r/2$; or
(c) $s \leq -1 \leq r \leq s/2$.

This is a simple consequence of Theorem 1. Namely, the function $f(x) = x^p$ is concave for $0 < p \leq 1$ and convex for $1 \leq p \leq 2$, or $-1 \leq p < 0$, while the functions $g(x) = x^{1/s}$ for $s \geq 1$ and $h(x) = -x^{1/s}$ for $s \leq -1$ are operator-monotone.Now, using these facts and substitutions $f(x) = x^{s/r}$, $x_i = C_i^r$ (or $f(x) = x^{r/s}$, $x_i = C_i^s$) we get (14).

Let us consider the cases (b) with $r = 1$ and (c) with $s = -1$. If $1 \geq s \geq \frac{1}{2}$, then

$$M_n^{[1]}(C;w) \geq M_n^{[s]}(C;w)$$

and if $-1 \leq r \leq -\frac{1}{2}$, then

$$M_n^{[-1]}(C;w) \leq M_n^{[r]}(C;w) .$$

Moreover, since for all $r \geq 1$, we have

$$M_n^{[r]}(C;w) \geq M_n^{[-1]}(C;w) ,$$

and for all $s \leq -1$,

$$M_n^{[s]}(C;w) \leq M_N^{[-1]}(C;w)$$

combining the previous inequalities we get that (23) is valid if either (c), or
(d) $r \geq 1 \geq s \geq \frac{1}{2}$; or
(e) $s \leq -1 \leq r \leq -\frac{1}{2}$.

Similary,substitutions $f(x) = x^{s/r}$, $C_i \to C_i^r$ or $f(x) = x^{r/s}$, $C_i \to C_i^s$ can be used in Theorems 2-10 as well. Here we shall only introduce three sorts of mixed means, i.e., we shall use these substitutions and Theorems 5,6 and 7.

$$M_n(s,t;k) :=$$

$$\left\{ \frac{1}{\binom{n-1}{k-1}} W_n \sum_{1 \leq i_1 < \ldots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) \left[\frac{w_{i_1} C_{i_1}^t + \cdots + w_{i_k} C_{i_k}^t}{w_{i_1} + \cdots + w_{ik}} \right]^{s/t} \right\}^{1/s},$$

$$\overline{M}(s,t;k) :=$$

$$\left\{ \frac{1}{\binom{n+k-1}{k-1}} W_n \sum_{1 \leq i_1 < \ldots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) \left[\frac{w_{i_1} C_{i_1}^t + \cdots + w_{i_k} C_{i_k}^t}{w_{i_1} + \cdots + w_{ik}} \right]^{s/t} \right\}^{1/s},$$

$$\underline{M}_n(s,t;k) := \left\{ \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1}^{n} w_{i_1}\ldots w_{i_k} \left(\frac{1}{k}(C_{i_1}^t + \cdots + C_{i_k}^t)\right)^{s/t} \right\}^{1/s}.$$

Remark. Note that the means $M_n(s,t;k)$ are only found in the literature in the discrete case and with $w_1 = \cdots = w_n = 1$ (see [4], pp. 191-193).

The following theorem is a consequence of Theorems 5,6 and 7.

Theorem 11. *Let A be an n-tuple of stricly positive operators, $w_i > 0$, $i = 1,...,n$. Then the following inequalities*

$$(15)\quad \begin{aligned} M_n^{[s]}(C;w) = M_n(s,r;1) \le \ldots \le M_n(s,r;k) \le \ldots \le M_n(s,r;n) \\ = M_n^{[r]}(C;w); \end{aligned}$$

$$(16)\quad M_n^{[s]}(C;w) = \overline{M}_n(s,r;1) \le \ldots \le \overline{M}_n(s,r;k) \le \ldots \le M_n^{[r]}(C;w);$$

$$(17)\quad M_n^{[s]}(C;w) = \underline{M}_n(s,r;1) \le \ldots \le \underline{M}_n(s,r;k) \le M_n^{[r]}(C;w);(1 \le k \le n),$$

are valid if either (i) $1 \le s \le r$; or (ii) $-r \le s \le -1$, or (iii) $s \le -1$, $r \ge s \ge 2r$; while the reverse inequalities are valid if either (iv) $r \le s \le -1$; or (v) $1 \le s \le -r$; or (vi) $s \ge 1$, $r \le s \le 2r$, are valid. For some related results see [17], *where generalizations of symmetric means are considered.*

4. Some inequalities for operator monotone functions. The following results is given in [1,p.29].

Let f be a continuous positive function on $(0,\infty)$, and A, B be positive operators. If f is operator-monotone, then

$$(18)\quad f(M_2^{[-1]}(A,B) \le M_2^{[-1]}(f(A),f(B)).$$

This is a simple consequence of the fact that the function $g(\lambda) = f(\lambda^{-1})^{-1}$ is operator-monotone and hence operator-concave.

Moreover, T.Ando [2] proved the following result:

Let f be a positive operator-monotone function on $(0,\infty)$|. The function $g(\lambda) = f(\lambda^{1/p})^p$ is operator monotone and hence operator concave if either $p \le -1$ or $p \ge 1$.

Remark. In fact, Ando considered matrices but, the proof is the same for operators.

For an n-tuple of operators $C = (C_1,...,C_n)$, we shall use the notation $f(C) = (f(C_1),...,f(C_n))$.

The following generalizations of (18) holds:

Theorem 12. *Let C be an n-tuple of stricly positive operators, let w be an n-tuple of positive numbers and let f be a positive operator-monotone function on $(0,\infty)$. If $p \ge 1$*

$$(19)\quad f(M_n^{[p]}(C;w)) \ge M_n^{[p]}(f(C);w)$$

while for $p \le -1$, the reverse inequality holds.

Proof. In both cases $p \ge 1$ and $p \le -1$, we have that the function $g(\lambda) = f(\lambda^{1/p})^p$ is operator-concave. Thus, theorem 1 gives

$$g\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i^p\right) \ge \frac{1}{W_n}\sum_{i=1}^{n} w_i g(C_i^p)$$

i.e.

$$f(M_n^{[p]}(C;w)^p \le \frac{1}{W_n}\sum_{i=1}^{n} f(C_i)^p . \tag{20}$$

If $p \ge 1$, the function $h(t) = t^{1/p}$ is operator monotone, so (20) gives (19). Moreover, if $p \le -1$, the function $f(t) = -t^{1/p}$ is operator monotone so that (20) gives the reverse inequality in (19).

Similarly, we can use Theorems 2-10 to obtain various related results. We shall only give some interpolations of (19) as consequences of theorems 5,6 and 7.

We introduce the following expressions:

$$g_{k,n}(p,f) := \left\{\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\le i_1<\dots<i_k\le n}(w_{i_1}+\dots+w_{i_k})f\left[\left(\frac{w_{i_1}C_{i_1}^p+\dots+w_{i_k}C_{i_k}^p}{w_{i_1}+\dots+w_{i_k}}\right)^{1/p}\right]^p\right\}^{1/p}$$

$$\overline{g}_{k,n}(p,f) = \left\{\frac{1}{\binom{n+k-1}{k-1}W_n}\sum_{1\le i_1\le\dots\le i_k\le n}(w_{i_1}+\dots+w_{i_k})f\left[\left(\frac{w_{i_1}C_{i_1}^p+\dots+w_{i_k}C_{i_k}^p}{w_{i_1}+\dots+w_{i_k}}\right)^{1/p}\right]^p\right\}^{1/p}$$

and

$$\underline{g}_{k,m}(p;f) = \left\{\frac{1}{W_n^k}\sum_{i_1,\dots,i_k=1}^{n} w_{i_1}\dots w_{i_k} f\left[\left(\frac{1}{k}(C_{i_1}^p+\dots+C_{i_k}^p)\right)^{1/p}\right]^p\right\}^{1/p} |>$$

The following theorem holds:

Theorem 13. *Let the conditions of* Theorem 12 *be satisfied. If $p \ge 1$, we have the following series of inequalities*

$$\begin{aligned} f(M_n^{[p]}(C;w)) = g_{n,n}(p,f) \ge \dots \ge g_{k+1,n}(p,f) \ge g_{k,n}(p,f) \ge g_{1,n}(p,f) \\ = M_n^{[p]}(f(C);w) \end{aligned} \tag{21}$$

$$\begin{aligned} f(M_n^{[p]}(C;w)) \ge \dots \ge \overline{g}_{k+1,n}(p,f) \ge \overline{g}_{k,n}(p,f) \ge \dots \overline{g}_{1,n}(p,f) \\ = M_n^{[p]}(f(C);w) \end{aligned} \tag{22}$$

(23) $f(M_n^{[p]}(C;w)) \geq \underline{g}_{k+1,n}(p,f) \geq \underline{g}_{k,n}(p,f) \geq \cdots \underline{g}_{1,n}(p,f) = M_n^{[p]}(f(C);w)$

If $p \leq -1$, the reverse inequalities in (21), (22) and (23) are valid.

REFERENCES

[1] Ando, T., *Topics on Operator Inequalities*, Sapporo 1978.

[2] Ando, T., *An inequality between symmetric function means of positive operators*, Acta Sci. Math. 45 (1983), 19–22.

[3] Bhaghwat, K.V. and R. Subramanian, *Inequalities between means of positive operators*, Math. Proc. Camb. Phil. Soc. 83 (1978), 393–401.

[4] Bullen, P.S., D.S. Mitrinović and D.M. Vasić, *Means and their inequalities*, D. Reidel Publ. Co., Dordrecht - Boston - Lancaster - Tokyo 1988.

[5] Dragomir, S.S., *An improvement of Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie 34 (82) (1990), 291–296.

[6] Dragomir, S.S., *An improvement of Jensen's inequality*, Mat. Bilten (Skopje) 15 (1991), 35–37.

[7] Mitrinović, D.S. and J.E. Pečarić, *Generalizations of the Jensen Inequality*, Sondend. Österr. Akad Wiss. 196 (1987), 21–25.

[8] Mond, B. and J.E. Pečarić, *Inequalities between means of positive operators*, (submitted for publication).

[9] Pećarić, J.E., F. Proschan and Y.L. Tong, *Convex function, partial orderings and statistical applications*, Academic Press Boston - San Diego - New York - London - Sydney - Tokyo and Toronto 1992.

[10] Pečarić, J.E., *Remark on an inequality of S.Gabler*, (submitted for publication).

[11] Pečarić, J.E. and D. Svrtan, *Refinement of Jensen's inequality with applications*, (submitted for publication).

[12] Pečarić, J.E. and S.S. Dragomir, *A refinement of Jensen inequality and applications*, Studia Univ. Babes-Bolyai, Math. 34 (1989), 15–19.

[13] Pečarić, J.E., *Remark on an interpolation of Jensen's inequality*, Prilozi MANU (Skopje), 11 (1990), 5–7.

[14] Pečarić, J.E., *Extension of an interpolation of Jensen's inequality*, Ibid. 15 (1991), 39–40.

[15] Pečarić, J.E. and S.S. Dragomir, *On an improvement of Jensen's inequality and applications*, (submitted for publication).

[16] Vasić, P.M. and J.E. Pećarić, *On Jensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.634-677 (1979), 50–54.

[17] Świętochowski, Z., *On the inequality of type $A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1}$*, Ann. Univ. Mariae Curie-Skłodowska Sect. A (1985), 159–163.

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