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On Functions Angularly Accessible in the Direction of the Imaginary Axis

O funkcjach kątowo osiągalnych w kierunku osi urojonej

Об углово-достижимых функциях в направлении мнимой оси

Introduction. Suppose that C denotes the complex plane, N is the set of natural numbers and $a \in \langle 0,1)$ is a fixed number. Now, let us assume the following notations:

$$A^-(w_0, a) = \left\{ w : \frac{\pi}{2}(3-a) \leqslant \arg(w-w_0) \leqslant \frac{\pi}{2}(3+a) \right\}$$

$$A^+(w_0, a) = \left\{ w : \frac{\pi}{2}(1-a) \leqslant \arg(w-w_0) \leqslant \frac{\pi}{2}(1+a) \right\}.$$

where $w_0 \in C$. A simply connected domain $D \neq C$ is called a-angularly accessible in the direction of the imaginary axis, if for every fixed point $w_0 \in C \backslash D$, either $A^+(w_0, a) \subset C \backslash D$, or $A^-(w_0, a) \subset C \backslash D$. The family of all such domains different from the whole plane C is denoted by T_a, while by $T_a(0)$ we denote the subfamily of T_a which consists of all domains containing the origin.

Let S_0 be the class of functions f analytic and univalent in the disc $E = E_1$, where $E_r = \{z : |z| < r\}$. The class of all functions $f \in S_0$ such that $f(E) \in T_a$ is denoted by I_a. T_0 is the family of domains convex in the direction of the imaginary axis, while I_0 is the well-known class of functions convex in the direction of the imaginary axis.

In this paper we give a necessary and sufficient condition for a function of S_0 to belong to I_a (Theorem 2). In the case $a = 0$ with an additional restriction such a theorem appears in a paper by M. S. Robertson [6], while without any restriction in a paper by W. Royster and M. Ziegler [7]. A different proof of results stated in the paper by W. Royster and M. Ziegler [7] is given in a paper by Cz. Burniak, Z. Lewandowski and J. Pituch [1].

Main results. We start with a density theorem for T_a. Our reasoning is a modification of that given in a paper by K. Ciozda [2] for the limit case $a = 0$.

Theorem 1. *Each domain D, $D \in T_a$ is a kernel in the sense of Carathéodory, of a decreasing sequence of domains obtained from the plane by removing a finite number of angles of the form $A^+(w, a)$ or $A^-(w, a)$.*

Proof. Let $n_0 \in N$ be a number such that $F_n = (C \setminus D) \cap \overline{E}_n \neq \emptyset$, where $\overline{E}_n = \{z : |z| \leq n\}$, $n \in \{n_0, n_0 + 1, \ldots\}$ and $D \in T_a$.

Since F_n is a compact set and $F_n \subset \overline{E}_n$, therefore there exists an ϵ – net, $\epsilon = \frac{1}{n}$, i.e. a set of such points $\{w_1, \ldots, w_s, v_1, \ldots, v_r\} \subset F_n$ that for each $w \in F_n$ there exists a number $l' \in \{1, 2, \ldots, s\}$ such that $|w - w_{l'}| < \frac{1}{n}$ or a number $l'' \in \{1, 2, \ldots, r\}$ such that $|w - v_{l''}| < \frac{1}{n}$. After a suitable change of order of points w_l, v_l we may choose positive integers k, l, $k \leq s$, $l \leq r$, so as to obtain the inclusion

$$\{w_1, \ldots, w_s, v_1, \ldots, v_r\} \subset \bigcup_{m=1}^{k} A^+(w_m, a) \cup \bigcup_{p=1}^{l} A^-(v_p, a).$$

Let $G_n = \bigcup_{m=1}^{k} A^+(w_m, a) \cup \bigcup_{p=1}^{l} A^-(v_p, a)$. It follows from the above construction that the distance of each point of the set F_n from the set G_n is less than $\frac{1}{n}$. In an analogous manner we form a set G_{n+1} such that $G_n \subset G_{n+1}$ and the distance of each point of the set F_{n+1} from the set G_{n+1} is less than $\frac{1}{n+1}$. In this way we define a decreasing sequence of domains $D_n = C \setminus G_n$. Since $D \subset D_n$ for $n \in \{n_0, n_0 + 1, \ldots\}$, $n_0 \in N$, therefore $D \subset \bigcap_{n=n_0}^{\infty} D_n$.

So, $D = \operatorname{Int} D \subset \operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n$. We will show that $D = \operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n$. Suppose that $D \neq \operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n$. Then there exists point $w_0 \in (\operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n) \setminus D$ and a number $\delta > 0$ such that $E(w_0, \delta) \subset \operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n$, where $E(w_0, \delta) = \{w : |w - w_0| < \delta\}$. Thus $E(w_0, \delta) \subset \bigcap_{n=n_0}^{\infty} D_n$, i.e. $\operatorname{dist}(w_0, C \setminus D_n) \equiv \operatorname{dist}(w_0, G_n) \geq \delta$, $n \in \{n_0, n_0 + 1, \ldots\}$. But $w_0 \in C \setminus D$, and consequently, for sufficiently large n, $w_0 \in F_n$ and $\operatorname{dist}(w_0, G_n) < \frac{1}{n}$. We may choose the number n in such a way that $\frac{1}{n} < \delta$ which leads to a contradiction because $\operatorname{dist}(w_0, G_n) \geq \delta$ for $n \in \{n_0, n_0 + 1, \ldots$. So $D = \operatorname{Int} \bigcap_{n=n_0}^{\infty} D_n$. Since

Int $\bigcap_{n=n_0}^{\infty} D_n$ is a kernel of sequence (D_n); our theorem follows.

Theorem 2. *Let f be a function non-constant and analytic in E. Then* $f \in I_a$ *if and only if there exist numbers* μ, ν, $0 \leqslant \mu \leqslant 2\pi$, $0 \leqslant \nu \leqslant \pi$, *such that*

$$|\arg\left\{-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})f'(z)\right\}| \leqslant (1-a)\frac{\pi}{2}, z \in E, \quad (1)$$

where $\arg(-i) = -\frac{\pi}{2}$.

Proof. 1. Let $f \in I_a$. We assume $f(0)=0$ i.e. $f(E) \in T_a(0)$. From Theorem 1 it follows that there is a sequence of domains containing the origin each of which is obtained from the plane by deleting a finite number of angles with measure $a\pi$, $a \in (0, 1\rangle$ whose bisectors are parallel to the imaginary axis. This sequence converges to the kernel $D = f(E)$. Let us first suppose that $D = f(E)$ is a domain which is obtained from the complex plane C by eliminating a finite numbers of angles of the form $A^+(w, a)$ or $A^-(w, a)$. We will approximate the domain D with an increasing sequence of polygons whose sides form with the real axis angles of absolute measure less than $(1-a)\frac{\pi}{2}$. Suppose first that the boundary of D is the sum of segments of half-lines which form sides of angles: $A^+(w_1, a) = A_1^+, \ldots, A^+(w_k, a) = A_k^+$ or $A^-(\nu_1, a) = A_1^-, \ldots, A^-(\nu_l, a) = A_l^-$, where $\operatorname{Re} w_1 < \ldots < \operatorname{Re} w_k, \operatorname{Re} \nu_1 < \ldots < \operatorname{Re} \nu_l, k, l, \in N$. There exists a number $M_1 > 0$ such that all the vertices of the polygon ∂D are contained in the strip $|\operatorname{Im} w| < M_1$. Let w_0', w_k' be common points of the line $\operatorname{Im} w = M_1$, and the left side of the angle A_1^+ and the right side of the angle A_k^+, respectively. The right side of the angle is a side in the right half-plane determined by the bisector of the angle. Analogously, let ν_0', ν_l' be common points of the line $\operatorname{Im} w = -M_1$ and the left side of the angle A_1^- and the right side of the angle A_l^-, respectively. Next, let w_j', $j = 1, 2, \ldots, k-1$ be common points of the right side of the angle A_j^+ and the left side of the angle A_{j+1}^+. Let ν_i', $i = 1, 2, \ldots, l-1$ be common points of the right side of the angle A_i^- and the left side of the angle A_{i+1}^-. Moreover, let P_1 be the common point of the straight line $\operatorname{Im} w = -M_1$ and a straight line containing the point w_0' and subtending with the positive direction of the real axis an angle of measure $(1-a)\frac{\pi}{2}$; let P_2 be the common point of the line $\operatorname{Im} w = -M_1$ and a straight line containing the point w_k' which subtends an angle of measure $(1+a)\frac{\pi}{2}$ with the positive direction of the real axis. If $C \setminus D$ does not contain angles $A^+(w, a)$ then we denote by P_1, P_2 common points of $\operatorname{Im} w = M_1$ and the straight line containing the point ν_0' which forms with the positive direction of the real axis an angle of measure $(1+a)\frac{\pi}{2}$ and the straight line containing the point v_l' which forms with the positive direction of the real axis an angle of measure $(1-a)\frac{\pi}{2}$, respectively. Let us form a polygonal line Γ_1 with vertices:

(i) $w_1 = v_1, v_1', v_2, v_2', \ldots, v_{l-1}', v_l = w_k, w_{k-1}', \ldots, w_1', w_1$, when D is bounded

(ii) $P_1, v_0', v_1, v_1', \ldots, w_1, w_0', P_1$, when D is unbounded from the left

(iii) $v_1, v_1', \ldots, v_l, v_l', P_2, w_k', w_k, \ldots, v_1$, when D is unbounded from the right

(iv) $P_1, v_0', v_1, v_1', \ldots, v_l, v_l', P_2, w_k', w_k, \ldots, w_1, w_0', P_1$, when D is unbounded from both sides

(v) $P_1, P_2, w_k', w_k, \ldots, w_1, w_0', P_1$, when none of the angles $A^-(w, a)$ is contained in $C \setminus D$

(vi) $P_1, v_0', v_1, v_1', \ldots, v_l, v_l', P_2, P_1$, when none of the angle $A^+(w, a)$ is contained in $C \setminus D$.

It follows from the above construction that $M_1 > 0$ may be chosen in such a way that the polygonal line Γ_1 is the boundary of a Jordan domain D_1, $D_1 \in T_a(0)$. We form a sequence (D_n), $D_n \in T_a(0)$, of domains constructed in the previously described manner while replacing M_1 by a sequence (M_n), $M_n \to +\infty$ for $n \to +\infty$ which is an increasing sequence of domains such that $\bigcup_{n=1}^{\infty} D_n = D = f(E)$. Hence D is a kernel of (D_n) in the sense of Carathéodory.

Let (f_n) be a sequence of functions $f_n \in S_0$ such that $\arg f_n'(0) = \arg f'(0)$, $f_n(E) = D_n$. It follows from the theorem of Carathéodory that $f_n \to f$ locally uniformly in E. There are real numbers $\psi_n, \theta_n, \psi_n \in \langle 0, 2\pi \rangle, \theta_n \in \langle 0, 2\pi \rangle, \psi_n - \theta_n > 0$ such that $f_n(e^{i\theta_n}) \in \Gamma_n$ and $\operatorname{Re} f_n(e^{i\theta_n})$ is the greatest, and $f_n(e^{i\psi_n}) \in \Gamma_n$ and $\operatorname{Re} f_n(e^{i\psi_n})$ is the least among the numbers in question. Assume that $\theta_n = \mu_n - v_n$, $\psi_n = \mu_n + v_n$, where $v_n \in (0, \pi)$, $\mu_n \in (0, 2\pi)$. At any point of $\Gamma_n = \partial f_n(E)$ (except for the vertices) we consider the normal vector. From the construction it follows that this vector forms with the positive direction of the real axis an angle of measure $a\frac{\pi}{2}$, or $\pi - a\frac{\pi}{2}$, or $\frac{\pi}{2}$ in the case 'upper part of Γ_n', and $\pi + \frac{a\pi}{2}$, or $2\pi - a\frac{\pi}{2}$, or $\frac{3}{2}\pi$ in the case 'lower part of Γ_n'. Points $f_n(e^{i\theta_n})$ and $f_n(e^{i\psi_n})$ uniquely determine the parts of Γ_n. Denote $f_n(e^{i\omega_j}) = w_j$, $j = 1, \ldots, k$; $f_n(e^{i\omega_j'}) = w_j'$, $j = 0, 1, \ldots, k$; $f_n(e^{i\gamma_m}) = v_m$, $m = 1, \ldots, l$; $f_n(e^{i\gamma_m'}) = v_m'$, $m = 0, 1, \ldots, l$ where $\psi_n \leq \gamma_0' < \gamma_1 < \gamma_1' < \ldots < \gamma_l \leq \gamma_l' \leq \theta_n \leq \omega_k' < \omega_k < \ldots < \omega_1' < \omega_1 < \omega_0' \leq \psi_n + 2\pi$.

At the points of ∂E where f_n admits analytic continuation (see G. M. Golusin [3]) we may consider the normal vector $\zeta f_n'(\zeta)$. Moreover, at ζ_0 (which corresponds to a vertex of Γ_n) the harmonic function $\arg(\zeta - \zeta_0)$ has a jump. Hence

$$\arg[-ie^{i\phi}f_n'(e^{i\phi})] = \begin{cases} \frac{\pi}{2}(1-a) \text{ for } \phi \in \bigcup_{j=1}^{k} (\omega_j', \omega_j) \cup (\omega_0', \psi_n + 2\pi) \\ -\frac{\pi}{2}(1-a) \text{ for } \phi \in \bigcup_{j=1}^{k} (\omega_j, \omega_{j-1}') \cup (\theta_n, \omega_k') \\ 0 \text{ for } \phi \in (\theta_n, \psi_n + 2\pi) \text{ when } k = 0 \end{cases} \tag{2}$$

for 'upper part of Γ_n' and

$$\arg[ie^{i\phi}f_n'(e^{i\phi})] = \begin{cases} \frac{\pi}{2}(1-a) \text{ for } \phi \in \bigcup_{i=1}^{l} (\gamma_{i-1}', \gamma_i) \\ 0 \text{ for } \phi \in (\psi_n, \gamma_0') \cup (\gamma_l', \theta_n) \\ -\frac{\pi}{2}(1-a) \text{ for } \phi \in \bigcup_{i=1}^{l} (\gamma_i, \gamma_i') \\ 0 \text{ for } \phi \in (\psi_n, \theta_n) \text{ when } l = 0 \end{cases}$$

for 'lower part of Γ_n'.

Let us consider the function

$$h(z;\mu,\nu) = \frac{ie^{-i\mu}z}{[1 - ze^{-i(\mu-\nu)}][1 - ze^{-i(\mu+\nu)}]}, \; z \in \overline{E}. \tag{3}$$

The boundary of the domain $h(E;\mu,\nu)$ is the sum of two half-lines contained in the imaginary axis which omit the origin. We easily examine that

$$\operatorname{Im} h(e^{i\phi};\mu,\nu) > 0 \text{ for } \phi \in (\mu-\nu, \mu+\nu)$$

$$\operatorname{Im} h(e^{i\phi};\mu,\nu) < 0 \text{ for } \phi \in (\mu+\nu, \mu-\nu+2\pi)$$

where $\mu \in (0, 2\pi)$, $\nu \in (0, \pi)$. Thus

$$\arg[h(e^{i\phi};\mu,\nu)] = \begin{cases} \frac{\pi}{2} = \arg i \quad \text{for } \phi \in (\mu-\nu, \mu+\nu) \\ -\frac{\pi}{2} = \arg(-i) \text{ for } \phi \in (\mu+\nu, \mu-\nu+2\pi). \end{cases} \tag{4}$$

From (2) and (4) it follows

$$\arg\frac{e^{i\phi}f_n'(e^{i\phi})}{h(e^{i\phi};\mu_n,\nu_n)}=\begin{cases}\frac{\pi}{2}(1-a) \text{ for} \\ \qquad \phi\in\bigcup_{j=1}^{k}(\omega_j',\omega_j)\cup\bigcup_{i=1}^{l}(\gamma_{i-1}',\gamma_i)\cup(\omega_0',\psi_n+2\pi) \\ 0 \text{ for } \phi\in(\psi_n,\gamma_0')\cup(\gamma_l',\theta_n) \\ 0 \text{ for } \phi\in(\psi_n,\theta_n) \text{ when } l=0 \\ 0 \text{ for } \phi\in(\theta_n,\psi_n+2\pi) \text{ when } k=0 \\ -\frac{\pi}{2}(1-a) \text{ for} \\ \qquad \phi\in\bigcup_{j=1}^{k}(\omega_j,\omega_{j-1}')\cup\bigcup_{i=1}^{l}(\gamma_i,\gamma_i')\cup(\theta_n,\omega_k').\end{cases}$$

Considering (3) we have:

$$\left|\arg\left\{-ie^{i\mu_n}(1-2e^{i\phi}e^{-i\mu_n}\cos\nu_n+e^{2i\phi}e^{-2i\mu_n})f_n'(e^{i\phi})\right\}\right|\le(1-a)\frac{\pi}{2} \tag{5}$$

for $\phi\in\langle\psi_n,\psi_n+2\pi\rangle\setminus\{\omega_1,\dots,\omega_k,\omega_0',\dots,\omega_k',\gamma_1,\dots,\gamma_l,\gamma_0',\dots,\gamma_l',\theta_n,\psi_n\}$. By Theorem 5 of the paper [5], p. 188, we have

$$\left|\arg\left\{-ie^{i\mu_n}(1-2ze^{-i\mu_n}\cos\nu_n+z^2e^{-2i\mu_n})f_n'(z)\right\}\right|\le(1-a)\frac{\pi}{2},\ z\in E. \tag{6}$$

Since $f_n\to f$ locally uniformly in E and the sequences (μ_n), (ν_n) are bounded, there exists a subsequence (n_k) such that $\mu_{n_k}\to\mu$, $\nu_{n_k}\to\nu$, $(k\to+\infty)$. From (6) with $n=n_k$ for $k\to+\infty$, we obtain

$$\left|\arg\left\{-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu+z^2e^{-2i\mu})f'(z)\right\}\right|\le(1-a)\frac{\pi}{2},\ z\in E. \tag{7}$$

We know that any domain of $T_a(0)$ can be approximated in the sense of Carathéodory by canonical domains (Theorem 1). Passing to the limit again we conclude that for $f\in S_0$, $f(0)=0$ such that $f(E)\in T_a(0)$ there exist numbers $\mu\in\langle 0,2\pi\rangle$, $\nu\in\langle 0,\pi\rangle$ which satisfy (7) and the first part of Theorem 2 follows.

2. Conversely, let $f(z)$, $f(0)=0$ be an analytic and non-constant function in E for which (7) holds.

a) If the sign of equality appears for some point $z\in E$, then by the maximum principle for harmonic functions we obtain

$$-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu+z^2e^{-2i\mu})f'(z)=ce^{\quad i(1-\alpha)\frac{\pi}{2}}.$$

Thus

$$f(z)=e^{\pm i(1-\alpha)\frac{\pi}{2}}\frac{c}{2\sin\nu}\ln\left[e^{-2i\nu}\frac{z-e^{-i(\mu+\nu)}}{z-e^{i(\mu-\nu)}}\right],\ f(0)=0. \tag{8}$$

For $\nu=0,\nu=\pi$ we must take the limit function of the form

$$f(z)=ie^{\pm i(1-\alpha)\frac{\pi}{2}}\frac{cze^{-i\mu}}{1-ze^{-i\mu}}.$$

Therefore $f(E)$ for $\nu\in(0,\pi)$, is a strip whose edges form with the imaginary axis an angle of measure $\alpha\dfrac{\pi}{2}$. For $\nu=0$ or $\nu=\pi$, $f(E)$ is a half-plane whose boundary forms an angle of measure $\alpha\dfrac{\pi}{2}$ with the imaginary axis; i.e. $f(E)\in T_\alpha(0)$ and the mapping is univalent.

b) Let us now assume that equality in (1) does not take place at any point in E. Thus

$$\left|\arg\left\{-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu+z^2e^{-2i\mu})f'(z)\right\}\right|<(1-\alpha)\frac{\pi}{2}. \tag{9}$$

It follows from the definition $h(\cdot\,;\mu,\nu)$ that the function H given by

$$H(z)=\int_0^z\frac{h(\zeta)}{\zeta}d\zeta=\frac{1}{2\sin\nu}\ln\left[e^{-2i\nu}\frac{z-e^{i(\mu+\nu)}}{z-e^{i(\mu-\nu)}}\right]$$

maps the disc E on the strip $\{w: A<\operatorname{Im}w<B\}$, where $-\infty\leqslant A<B\leqslant+\infty$. For every fixed $t\in(A,B)$ let us consider a straight line $L_t: w\equiv w(s)=s+ti$, $s\in(-\infty,+\infty)$. $H^{-1}(L_t)$ is a Jordan arc : $z_t=z_t(s)=H^{-1}(s+ti)$ contained in E with end-points at $e^{i(\mu-\nu)}$ and $e^{i(\mu+\nu)}$, respectively. Hence $H(z_t(s))=s+ti$ and

$$H'(z_t(s))=\frac{1}{\dfrac{d}{ds}z_t(s)}. \tag{10}$$

The condition (9) is equivalent to

$$| \arg \frac{f'(z)}{H'(z)} | < (1-a)\frac{\pi}{2}, \quad z \in E. \tag{11}$$

From (10) and (11) we obtain

$$| \arg \frac{d}{ds} f(z_t(s)) | < (1-a)\frac{\pi}{2}, \quad s \in (-\infty, +\infty). \tag{12}$$

Hence a tangent vector to the curve $z_t(s)$ forms with the positive direction of the real axis an angle larger than $-(1-a)\frac{\pi}{2}$ and simultaneously smaller than $+(1-a)\frac{\pi}{2}$. From convexity of H and from (11) it follows that f is a close-to-convex function, hence univalent (see W. Kaplan [4]). Therefore, if t varies from A to B, then the curves $z = z_t(s)$ have end-points $e^{i(\mu+\nu)}, e^{i(\mu-\nu)}$ in common only and they sweep out the disc E. Hence $f(D(t_1, t_2)) \in T_a$, where $t_1 \neq t_2$, $t_1, t_2 \in (A, B)$; $D(t_1, t_2)$, $D(t_1, t_2) \subset E$, denotes a domain bounded by the arcs $z = z_{t_1}(s)$, $z = z_{t_2}(s)$, $s \in (-\infty, +\infty)$. Hence $f(E) \in T_a$, i.e. $f \in I_a$.

In the second part of our proof we have exploited some ideas from the paper by Cz. Burniak, Z. Lewandowski and J. Pituch [1]. The proof is completed.

Theorem 3. *If $f \in I_a$, $0 \leq a \leq 1$, and $f(z) = z + a_2z^2 + \dots$, $a_2 \neq 0$, there exist numbers μ, ν; $a\frac{\pi}{2} \leq \mu \leq (2-a)\frac{\pi}{2}$, $0 \leq \nu \leq \pi$ such that*

$$| a_2 - e^{-i\mu} \cos \nu | \leq (1-a) | \cos \frac{2\mu - \pi}{2(1-a)} | . \tag{13}$$

Proof. By Theorem 2 there exist numbers μ, ν; $a\frac{\pi}{2} \leq \mu \leq (2-a)\frac{\pi}{2}$, $0 \leq \nu \leq \pi$ such that

$$| \arg \left\{ -ie^{i\mu}(1 - 2ze^{-i\mu} \cos \nu + z^2 e^{-2i\mu}) f'(z) \right\} | \leq (1-a)\frac{\pi}{2}, \quad z \in E.$$

Put $F(z) = -ie^{i\mu}(1 - 2ze^{-i\mu} \cos \nu + z^2 e^{-2i\mu}) f'(z)$. Thus the condition (1) has the form

$$| \arg [F(z)]^{\frac{1}{1-a}} | \leq \frac{\pi}{2},$$

which implies $\operatorname{Re} [F(z)]^{\frac{1}{1-a}} \geq 0$. Therefore, there is a function p ($\operatorname{Re} p(z) > 0$, $p(0) = 1$), such that

$$F(z) = [\cos \frac{2\mu - \pi}{2(1-a)} p(z) + i \sin \frac{2\mu - \pi}{2(1-a)}]^{1-a}$$

and consequently

$$f'(z)=\frac{\left[\cos\frac{2\mu-\pi}{2(1-a)}\,p(z)+i\sin\frac{2\mu-\pi}{2(1-a)}\right]^{1-a}}{-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu+z^2e^{-2i\mu})}. \tag{14}$$

which gives

$$p'(0)=\frac{2(a_2-e^{-i\mu}\cos\nu)}{(1-a)\cos\frac{2\mu-\pi}{2(1-a)}\exp\left[\frac{-ia(2\mu-\pi)}{2(1-a)}\right]}.$$

Since $|p'(0)|\leq 2$, we get $|a_2-e^{-i\mu}\cos\nu|\leq(1-a)\left|\cos\frac{2\mu-\pi}{2(1-a)}\right|$. This proves our statement.

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STRESZCZENIE

W pracy tej rozważa się klasę funkcji kątowo osiągalnych w kierunku osi urojonej. Podane są warunki konieczne i dostateczne na to, by funkcja należała do tej klasy.

РЕЗЮМЕ

В этой работе рассуждается класс углово-достижимых функций в направлении мнимой оси. Даны необходимые и достаточные условия для принадлежности функции к этому классу.