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On a Probability Distribution and Some Related Stochastic Processes

O rozkładzie prawdopodobieństwa i pewnych związanych z nim procesach stochastycznych

О распределениях и моментах порядковых статистик для выробки случайного объема

1. Introduction. An important part of probability theory is devoted to the study of finite and measurable functions, the random variables. Many of the most important problems concerning random variables can be expressed in terms of distribution functions. The method of classical analysis provide an efficient approach to problems of this sort.

It is frequently advisable to consider, instead of distribution functions, characteristic functions which are the Fourier transforms of distribution functions. It is now universally recognized that characteristic functions are the most powerful tools for the investigation of distribution functions. The uniqueness theorem, the convolution theorem, and the continuity theorem are the most significant theorems which describe the connections between characteristic functions and distribution functions. These properties account for the importance of characteristic functions in the theory of probability [4].

The present paper is devoted to the characteristic function

$$\phi(u) = \exp\left\{\lambda \int_0^u \frac{e^{iy} - 1}{y}\, dy\right\}, \quad -\infty < u < \infty \tag{1.1}$$

where $\lambda > 0$, for its mathematical interest and its applications [2].

The paper is organized as follows. In section two a characterization of $\phi(u)$ is given. This characterization deals with the distribution of stochastic integrals of continuous homogeneous processes with independent increments. Furthermore in this section the distribution function $F(x)$ of $\phi(u)$ is studied. Some properties of the distribution function $F(x)$ are established in terms of $\phi(u)$ since the explicit form of $F(x)$ is very complicated. Section three is devoted to the applications of $\phi(u)$. It is shown that $\phi(u)$ can be useful in certain industrial processes.

2. A Stochastic Integral. We shall now present some definitions and some results concerning stochastic processes and stochastic integrals. Let I be any bounded interval.

A stochastic process $\{X(t),\ t \in I\}$ is said to be a homogeneous process with independent increments if the distribution of the increment $X(t+h) - X(t)$ depends only on h but is independent of t and if the increments over non-overlapping intervals are independent. A stochastic process $\{X(t),\ t \in I\}$ is said to be continuous in probability if $\lim_{h \to 0} P[\,|X(t+h) - X(t)| > \epsilon\,] = 0$ for any $\epsilon > 0$.

Let $\{X(t),\ t \in I\}$ be a homogeneous and continuous in probability process with independent increments and denote the characteristic function of the increment $X(t+h) - X(t)$ by $\phi(u, h)$. It is well known that $\phi(u, h)$ is infinitely divisible and that $\phi(u, h) = [\,\phi(u, 1)]^h$. It can be shown that the stochastic integral

$$Y = \int_0^1 t\,dX(t)$$

exists in the sense of convergence in probability and the characteristic function $\phi(u)$ of Y is given by

$$\phi(u) = \exp\left\{\frac{1}{u}\int_0^u \log\phi(y, 1)\,dy\right\} \tag{2.1}$$

(see [5]) we prove the following theorem.

Theorem 1. *Let $\{X(t),\ t \in I\}$ be a homogeneous and continuous in probability process with independent increments. Let $X_1, X_2, \ldots$ be independent identically distributed random variables having the uniform distribution in $[0, 1]$, and N a Poisson random variable independent of $\{X(t),\ t \in I\}$ and $X_1, X_2, \ldots$ Then*

$$X(t+1) - X(t) \overset{d}{=} \int_0^1 t\,dX(t) + \sum_{i=1}^{N} X_i\,, \tag{2.2}$$

where $\overset{d}{=}$ means equality in distribution, if and only if $\phi(u, 1) = \exp\left\{\lambda \int_0^u \frac{e^{iy} - 1}{y}\,dy\right\}$.

Proof. Using characteristic functions in (2.2) we can write

$$\phi(u, 1) = \exp\left\{\frac{1}{u}\int_0^u \log\phi(y, 1)\,dy\right\}\exp\left\{\lambda\left(\frac{1}{u}\int_0^u e^{iy}\,dy - 1\right)\right\} \tag{2.3}$$

By taking logarithms in (2.3) and differentiating we have the differential equation

$$\phi'(u, 1) = \lambda\,\frac{e^{iu} - 1}{u}\,\phi(u, 1) \tag{2.4}$$

with the condition $\phi(0, 1) = 1$. Therefore

$$\phi(u, 1) = \exp\left\{\lambda \int_0^u \frac{e^{iy} - 1}{y}\,dy\right\}$$

Let X be the random variable which corresponds to the distribution function $F(x)$. An explicit but complicated expression of $F(x)$ is obtained by Takacs (1955). An asymptotic expression for $P[X > x]$ as $x \to \infty$ is obtained by De Brujn (1951). We close this section with the establishment of certain properties of $F(x)$ in terms of $\phi(u)$. We start with an observation about the unimodality of $F(x)$.

The characteristic function $\phi(u)$ belongs to the class of self-decomposable characteristic functions [1]. Theorem 1 of [9] implies the unimodality of $F(x)$. Below we establish a property for an infinite convolution of distribution functions having the same form with $F(x)$ and the connection of $F(x)$ with the Poisson distribution.

Theorem 2. *Let $\{P_n : n = 0, 1, \dots\}$ be a distribution on the non-negative integers with $P_n \geqslant P_{n+1}$ and $\gamma(u)$ its characteristic function. Then:*

(i) $\delta(u) = \exp\left\{\lambda \int_0^u \frac{e^{iy} - 1}{y} \gamma(y)\, dy\right\}$ *is the characteristic function of an infinite convolution of distribution functions having the same form with $F(x)$.*

(ii) *there exists a sequence of independent Poisson random variables $\{X_n : n = 1, 2, \dots\}$, with $E(X_n) = \lambda/n$, such that the sequence $\{S_n = [X_1 + 2X_2 + \dots + nX_n]/n : n = 1, 2, \dots\}$ converges in distribution to the random variable X.*

Proof. (i) Since $P_n \geqslant P_{n+1}$ we can write

$$\gamma(u) = \sum_{n=0}^{\infty} P_n e^{inu} = \sum_{n=1}^{\infty} q_n (1 - e^{inu})/(1 - e^{iu})$$

where $q_n = P_{n-1} - P_n \geqslant 0$. Hence

$$\prod_{n=1}^{\infty} \exp\left\{\lambda q_n \int_0^u \frac{e^{iny} - 1}{y}\, dy\right\} = \exp\left\{\lambda \int_0^u \frac{e^{iy} - 1}{y} \gamma(y)\, dy\right\} \tag{2.5}$$

Let $F_n(x)$ be the distribution function which corresponds to the characteristic function $\exp\left\{\lambda q_n \int_0^u \frac{e^{iny} - 1}{y}\, dy\right\}$. From (2.5) we conclude that $\delta(u)$ is the characteristic function of the infinite convolution $\prod_{n=1}^{\infty}{}^{*} F_n(x)$.

(ii) Using the partition $\{0, 1/n, 2/n, \dots, n/n\}$ of the interval $[0, 1]$ we can write

$$\phi(u) = \exp\left\{\lambda \int_0^1 \frac{e^{iuy} - 1}{y}\, dy\right\} = \lim_{n \to \infty} \prod_{k=1}^{n} \exp\left\{\frac{\lambda}{k}(e^{iku/n} - 1)\right\}.$$

Let X_k be the random variable whose characteristic function is $\exp\{\lambda/k\, (e^{iu} - 1)\}$. From the well-known continuity theorem we conclude that the sequence $\{S_n : n = 1, 2, \dots\}$ converges in distribution to the random variable X [6].

3. Applications of the Distribution. In this section it is shown that the characteristic function $\phi(u)$ can be useful in certain industrial processes.

(i) Total Replacement Cost. Let $X_1, X_2, \dots$ represent the lifetimes of items that are successively placed in service, the next item commencing service immediately following the failure of the previous one. We stipulate that $\{X_n : n = 1, 2, \dots\}$ are independent random variables with common distribution function $F(x) = 1 - \exp(-\lambda x), x > 0$. Starting with a new item at time $T_0 = 0$ we will have a replacement at each of the instant $T_1 = X_1, T_2 = X_1 + X_2, \dots$

We suppose that each replacement has a constant cost, which we suppose equal to 1. The present value at time 0, of the total replacement cost, which will be denoted by C, is

$$C = \sum_{n=1}^{\infty} \left\{ \exp -p \sum_{k=1}^{n} X_k \right\}, \tag{3.1}$$

where $p > 0$ is the rate of interest. Dall' Aglio [2] proves that the characteristic function of C is $[\phi(u)]^{1/p}$.

(ii) Total Manufacturing Cost. N units of a product are to be produced on a machine tool in a mass production plant. The set-up cost C_s, the machine time t_i, for the ith unit and the cost per unit machine time C_m are non-negative statistically independent random variables. The total manufacturing cost C is given by

$$C = C_s + C_m T \tag{3.2}$$

where $T = t_1 + t_2 + \ldots + t_N$. It is desired to derive the probability distribution of C, under the following assumptions:

a) C_s follows the degenerate probability distribution, and its characteristic function is e^{iuq}, where q positive constant

b) C_m follows the uniform probability distribution and its characteristic function is $(e^{iu} - 1)/iu$.

c) the probability distribution of T is not known but $C \stackrel{d}{=} T$, where $\stackrel{d}{=}$ means equality in distribution.

Using characteristic functions in (3.2) we can write

$$\beta(u) = \frac{e^{iqu}}{u} \int_0^u \beta(y)\, dy \tag{3.3}$$

where $\beta(u)$ is the characteristic function of C.

By multiplying both sides of (3.3) by ue^{-iqu} and differentiating we get $\beta(u) = e^{iqu}\,[\phi(qu)]^{1/\lambda}$.

c) Stock of Certain Objects. Consider the stochastic difference equation

$$Y_n = X_n Y_{n-1} + Z_n, \quad n = 1, 2, \ldots$$

where the pairs (X_n, Z_n) are independent, identically distributed R^2- valued random variables. This equation arises in various disciplines, for example economics, physics, nuclear technology, biology and sociology. In all applications Y_n represent a stock of certain objects at time n, Z_n the quantity that is added just before time n (or taken away in case $Z_n < 0$) and the factor X_n indicates the intrinsic decay or increase of the stock Y_{n-1} between times $n - 1$ and n.

We mention an example from nuclear technology. In this example, due to Uppuluri, Feder and Shenton [8] Y_n represents a stock of radioactive material at time n, Z_n the quantity added just before time n and X_n the natural decay of radioactivity. In this example $Z_n = 1$ with probability one, and X_n are independent, identically distributed uniform variables on the interval [0, 1]. Under these conditions Y_n converges in distribution to a random variable Y whose characteristic function is $e^{iu}\,[\phi(u)]^{1/\lambda}$.

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STRESZCZENIE

W pracy podano dwa twierdzenia charakteryzujące rozkład prawdopodobieństwa o funkcji charakterystycznej

$$\phi(u) = \exp\left\{\lambda \int_0^u [(e^{iu} = 1)/y]\,dy\right\}, \quad -\infty < u < \infty$$

gdzie $\lambda > 0$ jest pewną stałą.

РЕЗЮМЕ

В этой работе дается две теоремы, характеризующие распределение вероятности, которой характеристическая функция имеет вид

$$\phi(u) = \exp\ \lambda \int_0^u [(e^{iy} = 1)/y]\,dy\ , \quad -\infty < u < \infty$$

где $\lambda > 0$ — некоторая константа.