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Łukasz KRUK and Wiesław ZIĘBA (Lublin)

On Almost Sure Convergence of Asymptotic Martingales

Abstract. The aim of this paper is to give a characterization of almost sure convergence for sequences of random variables, which do not necessarily have first moments. An example of such characterization was given in [5], where a notion of a D_v-amart was introduced. In this work we show that every D_v-amart converges a.s. A proof of this fact can be also found in [5], although it was not mentioned by the author. In the second part of this paper we give proofs of conditional lemmas of Borel-Cantelli. Then we use them to prove a conditional version of the Kolmogorov's strong law of large numbers, in which assumption that expectations exist was reduced.

Let (Ω, A, P) be a probability space, $\{F_n, n \geq 1\}$ an increasing (i.e. $F_n \subset F_{n+1}$) sequence of sub-σ-fields of a σ-field A. We denote by T a set of all bounded stopping times ($P(\tau < M) = 1$, where M depends on τ). A sequence $\{X_n, n \geq 1\}$ is adapted to $\{F_n, n \geq 1\}$ if X_n is F_n-measurable for every $n \geq 1$ .amarts can be found in [6], [7]. In the definition of an amart we assume that

$$E|X_n| < \infty\,, \tag{1}$$

where $E(\cdot)$ denotes the expectation.

In [5] a definition of a D_v-amart was given, with omitted assumption (1) and unchanged properties of an amart.

In [11] a notion of a conditional amart was introduced. Properties of conditional amarts were examined in [10] and [11]. In the definition of a conditional amart the assumption (1) was replaced by a weaker one.

Let $\tau \in T$, i.e. $[\tau = n] \in F_n$ for $n \geq 1$ and $P[\tau \leq M] = 1$ for some M (depending on τ).

The definition of a conditional expectation with respect to a σ-field $F \subset A$ of a nonnegative random variable can be found in [9]. Let $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$, then $X = X^+ - X^-$. If $\min(E^F X^+, E^F X^-) < \infty$ a.s., then $E^F X = E^F X^+ - E^F X^-$. A fact that $\max(E^F X^+, E^F X^-) < \infty$ a.s. is equivalent to $E^F|X| < \infty$ a.s. If one of these conditions holds, we write $X \in L^1_F$. Similarly, we write $X \in L^2_F$ if $E^F X^2 < \infty$ a.s.

Definition 1 [11]. An adapted sequence $\{X_n, n \geq 1\}$ of random variables is called a conditional amart (with respect to a sub-σ-field F), if

1. $X_n \in L^1_F, n \geq 1$,

2. A net $L(E^{F}X_{\tau}, X), \tau \in T$, converges to zero for some random variable X, where L denotes the Levy-Prokhorov metric.

If $F = \{\emptyset, \Omega\}$, we obtain the definition of an amart.

In general, the assumption 1. is weaker than $X_n \in L^1$ $(E|X_n| < \infty)$, $n \geq 1$.

Let I denote a class of continuous decreasing functions v defined on $(0, \infty)$ and satisfying the following conditions:

a) $\lim_{\lambda\to\infty} v(\lambda) = 0$, $\lim_{\lambda\to 0} v(\lambda) = +\infty$,

b) There exists $\alpha \in (0,1)$ such that $\sup_{\lambda>0} \frac{v(\alpha\lambda)}{v(\lambda)} = C_\alpha < \infty$. [2]

Let

$$\|X\|_v = \inf\{\gamma : \sup_{\lambda>0} P[|X| > \lambda\gamma]/v(\lambda) < \gamma\} \tag{2}$$

and let D_v denote a set of random variables such that $X \in D_v$ iff $\lim_{\lambda\to\infty} \frac{P[|X|>\lambda]}{v(\lambda)} = 0$. If $X \in D_v$, then $\|X\|_v < \infty$ and a metric space (D_v, ρ) is complete and separable, where $\rho(X,Y) = \|X - Y\|_v$. Proofs of these facts can be found in [4].

In [5] a notion of a D_v-amart was introduced.

Definition 2. An adapted sequence $\{X_n, n \geq 1\}$ of r.v.s is called a D_v-amart iff

3. $X_n \in D_v, n \geq 1$, for some function $v \in I$,
4. for every $\epsilon > 0$ there exists $\tau_0 \in T$ such that $\|X_\tau - X_\sigma\| < \epsilon$ for $\tau, \sigma \in T, \tau, \sigma \geq \tau_0$ a.s.

Let $r(X,Y) = \inf\{\epsilon > 0 : P[|X - Y| > \epsilon] < \epsilon\}$ denote the Ky-Fan metric.

Theorem 1. *There exists a constant V_0 such that $r(X,Y) \leq V_0 \|X - Y\|_v$.*

Proof. From the definition of $\|X\|_v$ we have

$$\forall\epsilon > 0 \quad \sup_{\lambda>0} \frac{P[|X - Y| > \lambda(\|X - Y\|_v + \epsilon)]}{v(\lambda)} \leq \|X - Y\|_v + \epsilon\,.$$

Thus for an arbitrary $\lambda > 0$ and $\epsilon > 0$

$$P[|X - Y| > \max(\lambda, v(\lambda))(\|X - Y\|_v + \epsilon)] \leq \max(\lambda, v(\lambda))(\|X - Y\|_v + \epsilon)\,.$$

Let $V_0 = \min_{\lambda>0}(\max(\lambda, v(\lambda)))$, then

$$P[|X - Y| > V_0 \|X - Y\|_v] \leq V_0 \|X - Y\|_v\,,$$

so

$$r(X,Y) \leq V_0 \|X - Y\|_v$$

and the proof is complete.

Corollaries.

1. If $\{X_n, n \geq 1\}$ is a sequence of random variables such that $\|X_n - X\|_v \to 0, n \to \infty$, for some r.v. X, then this sequence converges in probability to X, i.e. $X_n \xrightarrow{P} X\,, n \to \infty$.

2. If a sequence $\{X_n, n \geq 1\}$ is a D_v-amart, then it satisfies a condition

$$\forall \epsilon > 0\ \exists \tau_0 \in T\ \forall \tau, \sigma \geq \tau_0 \text{ a.s. } r(X_\tau, X_\sigma) < \epsilon . \tag{3}$$

We shall show that (3) implies almost sure convergence of $\{X_n, n \geq 1\}$.

Theorem 2. *If $\{X_n, n \geq 1\}$ is a sequence satisfying (3), then for every sequence $\{\tau_n, n \geq 1\}$ such that $\tau_n \in T, n \geq 1$, and $\tau_n \xrightarrow{P} \infty$, $n \to \infty$, $X_{\tau_n} \xrightarrow{P} X$, $n \to \infty$, for some r.v. X.*

Proof. If a sequence satisfies (3), then it satisfies also the Cauchy's condition. Completeness of the space (Φ, r) (where Φ denotes a set of random variables) implies the existence of a r.v. X such that $r(X_n, X) \to 0, n \to \infty$.

Let $\{\tau_n, n \geq 1\}$ be an arbitrary sequence satisfying the following conditions: $\tau_n \in T, n \geq 1$, and $\tau_n \xrightarrow{P} \infty$. Then

$$\forall k \in N\ \exists n_k\ \forall n > n_k\ P[\tau_n < k] < \frac{1}{2^k} .$$

We may assume that the sequence $\{n_k, k \geq 1\}$ is increasing. Denote $A_k = \{n : n_{k-1} < n \leq n_k\}$, where $n_0 = 0$. We have $N = \cup_{k=1}^{\infty} A_k$. Define a sequence $\{\tau'_n, n \geq 1\}$ in the following way: if $n \in A_k$, then $\tau'_n = \tau_n$ if $\tau_n \geq k$ and $\tau'_n = k$ if $\tau_n < k$. It is easy to see that $P[\tau'_n \neq \tau_n] < \frac{1}{2^k}$ for $n \in A_k$, thus $P[\tau'_n \neq \tau_n] \to 0, n \to \infty$.

It is easy to see that $X_{\tau_n} \xrightarrow{P} X$, $n \to \infty$, iff $X_{\tau'_n} \xrightarrow{P} X, n \to \infty$, because

$$r(X_{\tau_n}, X) \leq r(X_{\tau_n}, X_{\tau'_n}) + r(X_{\tau'_n}, X) \leq P[\tau'_n \neq \tau_n] + r(X_{\tau'_n}, X)$$

and similarly

$$r(X_{\tau'_n}, X) \leq P[\tau'_n \neq \tau_n] + r(X_{\tau_n}, X) .$$

The condition (3) implies $X_{\tau'_n} \xrightarrow{P} X$, $n \to \infty$. This completes the proof.

Theorem 3. *Let $\{X_n, n \geq 1\}$ satisfy (3). Then this sequence converges almost surely to some random variable X.*

Proof. The space (Φ, r) is complete and therefore there exists a random variable X such that $r(X_n, X) \to 0, n \to \infty$. Let $X^* = \limsup X_n$ and $X_* = \liminf X_n$. Then (see [1]) there exist sequences of bounded stopping times $\{\tau_n, n \geq 1\}$ and $\{\sigma_n, n \geq 1\}$ such that $\tau_n \geq n, \sigma_n \geq n$, $\lim X_{\tau_n} = X^*$ a.s. and $\lim X_{\sigma_n} = X_*$ a.s. Obviously

$$r(X^*, X_*) \leq r(X^*, X_{\tau_n}) + r(X_{\tau_n}, X_{\sigma_n}) + r(X_{\sigma_n}, X_*) \to 0,\ n \to \infty,$$

by (3), so $r(X^*, X_*) = 0$ and the proof is complete.

Corollary . *Every D_v-amart converges a.s.*

Indeed, every D_v-amart satisfies the condition (3), so it converges a.s.

A proof of this fact follows also from (3) and the second part of theorem 1 [5]. The converse to the above theorem can also be proved.

Theorem 4. *Let $\{X_n, n \geq 1\}$ be an adapted sequence of random variables. If $\{X_n\}$ converges a.s. to some r.v. X, then it is a D_v- amart for some function $v \in I$.*

Proof. Let $Y = \sup|X_n|$. By hypothesis, $Y < \infty$ a.s. There exists a continuous, decreasing function v defined on $(0, \infty)$ satisfying the conditions a) and b) such that $Y \in D_v$ (see [4], [5]).

Obviously $|X_n| \leq Y$ a.s. and $|X| \leq Y$ a.s., so X_n and Y belong to D_v. Similarly for an arbitrary finite stopping time τ $X_\tau \in D_v$. Let τ and σ be finite stopping times. $|X_\tau - X_\sigma| \leq 2Y$, so, by b)

$$\lim_{\lambda\to\infty} \frac{P[|X_\tau - X_\sigma| > \lambda]}{v(\lambda)} \leq \lim_{\lambda\to\infty} \frac{P[2Y > \lambda]}{v(\lambda)} = \lim_{\lambda\to\infty} \frac{P[Y > \frac{\lambda}{2}]}{v(\lambda)}$$

$$\leq \lim_{\lambda\to\infty} C_\alpha^m \frac{P[Y > \frac{\lambda}{2}]}{v(\frac{\lambda}{2})} = C_\alpha^m \lim_{\lambda\to\infty} \frac{P[Y > \lambda]}{v(\lambda)} = 0\,,$$

where m is so large natural number that $\alpha^m < \frac{1}{2}$. Thus $X_\tau - X_\sigma \in D_v$.

Let $\eta > 0$ be an arbitrary constant. We want to find $n \in N$ such that for all bounded stopping times $\tau, \sigma \geq n$ a.s.

$$\frac{P[|X_\tau - X_\sigma| > \lambda\eta]}{v(\lambda)} < \frac{\eta}{2} \tag{4}$$

for every $\lambda > 0$, because it implies $\|X_\tau - X_\sigma\| \leq \epsilon$, what completes the proof.

It is obvious that (4) holds for $v(\lambda) > \frac{2}{\eta}$. Because $\lim_{\lambda\to\infty} v(\lambda) = \infty$ and v is decreasing, there exists a_η such that $v(\lambda) > \frac{2}{\eta}$ for $0 < \lambda < a_\eta$. Take $m \in N$ such that $\alpha^m < \eta$, where α fulfils the condition b). Thus, by b), $v(\lambda\eta) \leq v(\lambda\alpha^m) \leq C_\alpha^m v(\lambda)$, thus

$$\frac{P[|X_\tau - X_\sigma| > \lambda\eta]}{v(\lambda)} \leq C_\alpha^m \frac{P[|X_\tau - X_\sigma| > \lambda\eta]}{v(\lambda\eta)},$$

what tends to zero as $\lambda \to \infty$ by the definition of D_v. Let us choose b_η so large that the right side of the last inequality is less than $\frac{\eta}{2}$ for $\lambda > b_\eta$. Thus (4) holds also for $\lambda > b_\eta$.

Now let $\lambda \in [a_\eta, b_\eta]$. $v(\lambda) \geq v(b_\eta) > 0$, so it is enough to find such n that for $\tau, \sigma \geq n$ a.s., $\tau, \sigma \in T$, $P[|X_\tau - X_\tau| > \lambda\eta] < \frac{\eta}{2} v(b_\eta)$. We have $P[|X\tau - X_\sigma| > \lambda\eta] \leq P[|X_\tau - X_\sigma| > a_\eta\eta]$. Because X_n converges almost surely to X , $\lim_{n\to\infty} P[\sup_{m,l\geq n} |X_m - X_l| > a_\eta\eta] = 0$. Let us choose n so large that $P[\sup_{m,l\geq n} |X_m - X_l| > a_\eta\eta] < \frac{\eta}{2} v(b_\eta)$. Obviously for all bounded stopping times $\tau, \sigma \geq n$ a.s. $P[|X_\tau - X_\sigma| > a_\eta\eta] < \frac{\eta}{2} v(b_\eta)$, what completes the proof.

The following theorem is also true.

Theorem 5. *If $\{X_n, n \geq 1\}$ is an adapted sequence of random variables converging a.s. to X, then there exists a sequence of disjoint sets $\{B_n, n \geq 1\}$ such that $B_n \in A, n \geq 1, P(\cup_{n=1}^\infty B_n) = 1$, $\{X_n, F_n, n \geq 1\}$ is a conditional amart with respect to a σ-field $F = \sigma(B_n, n \geq 1)$ and $E^F \sup_{n\geq 1} |X_n| < \infty$.*

Proof. $\sup|X_n| < \infty$ a.s. since X_n converges to X a.s. Let $A_k = [|X_n| < k,\ n \geq 1]$, $k \geq 1$. Obviously $A_1 \subset A_2 \subset \ldots$ and $P(\cup_{n=1}^{\infty} A_n) = 1$. If $B_1 = A_1$ and $B_n = A_n \backslash A_{n-1}$ for $n \geq 2$, then $\{X_n, F_n, n \geq 1\}$ is a conditional amart with respect to a σ-field $F = \sigma(B_n, n \geq 1)$ and $E^F \sup_{n\geq 1} |X_n| < \infty$ a.s. Indeed, $\sup|X_n| \leq \sum_{k=1}^{\infty} kI_{B_k}$, thus $E^F \sup|X_n| \leq E^F \sum_{k=1}^{\infty} kI_{B_k} < \infty$ a.s. and so $\sup|X_n| \in L_F^1$. For every k $|X_k| \leq \sup|X_n|$, so $X_k \in L_F^1$.

Let $\epsilon > 0$ and let $m \in N$ be so large that $P(\cup_{k=1}^{m} B_k) > 1 - \epsilon$. Let $n_1 > m$ be so large that for every $k = 1, \ldots m$ such that $P(B_k) > 0$ and for every $\tau \geq n_1$ a.s.

$$|E^F(X_\tau - X)I_{B_k}| \leq \frac{1}{P(B_k)} \int_{B_k} |X_\tau - X|\, dP$$

$$\leq \frac{1}{P(B_k)} \int_{B_k} \sup_{n \geq n_1} |X_n - X|\, dP < \epsilon$$

(it is possible by the Lebesgue dominated convergence theorem). Thus $P[|E^F X_\tau - E^F X| > \epsilon] < \epsilon$, so $r(E^F X_\tau, E^F X) \leq \epsilon$ if $\tau \geq n_1$ a.s. $L(X,Y) \leq r(X,Y)$ for any r.v.s X, Y and so $L(E^F X_\tau, E^F X) \leq \epsilon$ if $\tau \geq n_1$. The proof is complete.

0.1. Conditional lemmas of Borel-Cantelli and conditional laws of large numbers. Now we shall give generalized lemmas of Borel-Cantelli. Moreover, we shall show how to generalize the Kolmogorov's strong law of large numbers weakening the condition (1).

Let F be a sub-σ-field of a σ-field A.

Lemma 1. *If $\{A_n, n \geq 1\}$ is a sequence of random events such that $\sum P(A_n|F) < \infty$ a.s., where $P(A|F) = E^F I_A$, $E = (\limsup A_n)^c = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^c$, then $P(E) = 1$.*

Proof. We shall show that $P(E^c) = 0$.

$$0 \leq P(E^c|F) = P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k|F) = \lim_{n\to\infty} P(\cup_{k=n}^{\infty} A_k|F)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k|F) = 0 \quad \text{a.s.}$$

Hence $P(E^c) = 0$ and $P(E) = 1$.

Let us remark that convergence of $\sum P(A_n|F)$ does not imply convergence of $\sum P(A_n)$.

Example 1. Let $(\Omega, A, P) = ([0,1], B([0,1]), \mu)$, where μ is the Lebesgue measure on the unit interval, $A_n = (0, \frac{1}{n})$, $n \geq 1$, and $F = \sigma(A_n, n \geq 1)$. It is easy to see that $\sum_{n=1}^{\infty} P(A_n|F) = \sum_{n=1}^{\infty} I_{A_n} < \infty$ a.s., but $\sum P(A_n) = \sum \frac{1}{n} = \infty$.

You can also prove a fact, which is, in some sense, a converse to the above.

Lemma 1*. *If $\{A_n, n \geq 1\}$ is a sequence of random events and $P(\limsup A_n) = 0$, then for every σ-field F such that $\sigma(A_n, n \geq 1) \subset F \subset A$ we have $\sum_{n=1}^{\infty} P(A_n|F) < \infty$.*

Let (Ω, A, P) be a probability space and F a nonempty sub-σ-field of A.

Definition 3. Events $B, C \in A$ are called F-independent, if $P(B \cap C|F) = P(B|F) \cdot P(C|F)$ a.s.

σ-fields G_1, G_2 are F-independent, if every two events $A_1 \in G_1$ and $A_2 \in G_2$ are F-independent.

Random variables X and Y are F-independent, if σ-fields generated by these variables are F-independent.

In such case if, in addition, $X, Y, XY \in L^1_F$, then $E^F XY = E^F X \cdot E^F Y$ a.s.

Let us remark that if X is F-measurable and Y is an arbitrary r.v., then X and Y are F-independent.

Lemma 2. *Let $\{A_n, n \geq 1\}$ be a sequence of F-independent events and let $A = \{\omega : \sum_{n=1}^{\infty} P(A_n|F)(\omega) = \infty\}$. Then $P(\limsup A_n) = P(A)$.*

Proof. Let $E = (\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k)^c = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^c$. Properties of conditional expectations imply

$$P(E|F) = \lim_{n\to\infty} P(\cap_{k=n}^{\infty} A_k^c|F) = \lim_{n\to\infty} (\lim_{k\to\infty} P(\cap_{i=n}^{\infty} A_i^c|F))$$

$$= \lim_{n\to\infty} (\lim_{k\to\infty} \prod_{i=n}^{k} P(A_i^c|F)) = \lim_{n\to\infty} \lim_{k\to\infty} [\prod_{i=1}^{k} (1 - P(A_i|F))]$$

$$= \lim_{n\to\infty} \prod_{i=n}^{\infty} (1 - P(A_i|F)) \leq \lim_{n\to\infty} \exp(-\sum_{i=n}^{\infty} P(A_i|F)) \quad \text{a.s.}$$

(the last inequality follows from an inequality $1 - x \leq \exp(-x)$ for $x \in [0,1]$). Thus for almost every $\omega \in A$ we have

$$0 \leq P(E|F)(\omega) \leq \lim_{n\to\infty} \exp(-\sum_{i=n}^{\infty} P(A_i|F)(\omega)) = 0 \qquad \text{a.s.}$$

Thus

$$P(E) = \int_{\Omega} P(E|F)\, dP = \int_{A} P(E|F)\, dP + \int_{A^c} P(E|F)\, dP \leq P(A^c)\,,$$

so $P(E^c) \geq P(A)$.

On the other hand, following the reasoning given in lemma 1, we state that on the set A^c only finitely many events from the sequence $\{A_n, n \geq 1\}$ hold, so $P(E^c) \leq P(A)$, q.e.d.

Theorem 6. *If G_1 and G_2 are F-independent σ-fields, then $\sigma(G_1, F)$ and G_2 are F-independent σ-fields as well.*

Definition 4. Let $X \in L^2_F$. A random variable $\sigma^2_F X$ defined by a formula $\sigma^2_F X = E^F(X - E^F X)^2$ will be called a conditional variance of X.

Similarly as in the case of independent r.v.s (see [3]) the following theorem may be proved.

Theorem 7. *Assume that*

$$\begin{matrix} X_{11} & X_{12} & \dots \\ X_{21} & X_{22} & \dots \\ \dots & \dots & \dots \end{matrix},$$

is a matrix of F-independent r.v.s and $F_i = \sigma(X_{i1}, X_{i2}, \ldots)$, $i = 1, 2, \ldots$. *Then the* σ*-fields* $F_1, F_2, \ldots$ *are F-independent.*

The above results lead us to a generalization of the well-known Kolmogorov's inequality.

Theorem 8. *If* $\{X_n, n \geq 1\}$ *is a sequence of F-independent r.v.s belonging to* L_F^2, *then for an arbitrary F-measurable r.v.* $\epsilon > 0$ *a.s. we have*

$$\epsilon^2 P[\max_{1\leq k\leq n} |S_k - E^F S_k| \geq \epsilon | F] \leq \sum_{k=1}^{n} \sigma_F^2 X_k \quad a.s. ,$$

where $S_n = X_1 + \ldots + X_n$.

This inequality implies the conditional Kolmogorov's strong law of large numbers.

Theorem 9. *If* $\{X_n, n \geq 1\}$ *is a sequence of F-independent r.v.s such that*

$$(*) \qquad \sum_{k=1}^{\infty} \frac{\sigma_F^2 X_k}{k^2} < \infty \quad a.s.$$

then

$$\frac{S_n - E^F S_n}{n} \to 0 \qquad a.s. \ as \ n \to \infty .$$

Definition 5. We say that r.v.s X, Y are identically F-distributed, if for every Borel set $B \subset R$ $P(X \in B|F) = P(Y \in B|F)$ a.s.

Theorem 10. *Let* $\{X_n, n \geq 1\}$ *be a sequence of F-independent, identically F-distributed r.v.s and let* $S_n = X_1 + \ldots + X_n$. *Then* $\frac{S_n}{n} \to Z$ *a.s. for some r.v. Z iff* $X_1 \in L_F^1$. *If this condition holds, then* $Z = E^F X_1$.

Example 2. Let $(\Omega, A, P) = ([0,1], B([0,1]), \mu)$, where μ is the Lebesgue measure, and let $F = \sigma([0, \frac{1}{2}], (\frac{1}{2}, 1])$. Let $X_n(\omega) = 1$ for $\omega \in [0, \frac{1}{2}]$ and $X_n(\omega) = -1$ for $\omega \in (\frac{1}{2}, 1]$. $\frac{S_n - E^F S_n}{n} = 0 \to 0$, but you cannot find real numbers A_n such that $\frac{S_n - A_n}{n} \to 0$ a.s.

Proofs of the above generalizations are similar to proofs of the corresponding well-known theorems.

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Authors' address:
Instytut Matematyki UMCS
Plac M. Curie Skłodowskiej 1
20–031 Lublin, Poland