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Subordination and Majorization for some Classes of Holomorphic Functions

Podporządkowanie a majoryzacja dla pewnych klas funkcji holomorficznych

Подчинение и мажорирование для некоторых классов голоморфических функций

We introduce the following notations:

$\mathbb{C}$ – complex plane,

$K_R = \{z \in \mathbb{C} : |z| < R\}$,

$H(D)$ – the class of all functions holomorphic in a domain D,

$B = \{\phi \in H(K_R) : |\phi(z)| \le 1 \quad \text{for} \quad z \in K_R\}$

$B_n = \{\phi \in B : \phi(z) = \beta_{n-1} z^{n-1} + \beta_n z^n + \dots\}, \quad n = 1,2,\dots,$

$\Omega = \{\omega \in H(K_R) : |\omega(z)| \le |z| \quad \text{for} \quad z \in K_R\}$,

$\Omega_n = \{\omega \in \Omega : \omega(z) = \alpha_n z^n + \alpha_{n+1} z^{n+1} + \dots\}, \quad n = 1,2,\dots$,

$N = \{F \in H(K_1) : F(0) = 0, \ F'(0) = 1\}$,

$N_n = \{f \in H(K_1) : f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots, \ a_1 \ge 0\}$,

$n = 1,2,\dots$.

We say that a function $f \in H(K_R)$ is subordinate to a function $F \in H(K_R)$ in a disc K_R and write

$$f \prec F \quad \text{in} \quad K_R$$

if there exists a function $\omega \in \Omega$ such that

$$f(z) = F(\omega(z)), \qquad \text{for} \quad z \in K_R.$$

We say that a function $f \in H(K_R)$ is majorized by $F \in H(K_R)$ in a disc K_R and write

$$f \ll F \qquad \text{in} \quad K_R$$

if there exists a function $\varphi \in B$ such that

$$f(z) = F(z)\,\varphi(z), \qquad \text{for} \quad z \in K_R.$$

Z. Lewandowski [3] has begun to study the relationships between majorization of functions in the unit disc K_1 and their subordination in some smaller disc K_r. Next Z. Lewandowski and the present author had generalized this problem. In papers [4,5] they had investigated a relationship between majorization of functions in K_1 and inclusion of the image domains of some concentric discs. A general method of solving this problem has been given in [4].

In this paper a relationship between subordination and inclusion the maps of some concentric discs is investigated in a case when f ranges over the class N_n, $(n \geqslant 2)$ and F ranges over some special class $S(m,M)$. The class $S(m,M)$ can be defined as follows:

DEFINITION 1. Let $m = m(r)$, $M = M(r)$, $(m(0) = M(0) = 0$, $m(r) \leqslant M(r))$, be two nonnegative and increasing functions for $r \in \langle 0,1)$. We say that a function $F \in S(m,M)$ if $F \in N$ and for $|z| = r < 1$ a following inequality

$$m(r) \leqslant |F(z)| \leqslant M(r) \tag{1}$$

holds.

We can write then

$$S(m,M) = \left\{ F \in N : \bigwedge_{|z|=r<1} m(r) \leqslant |F(z)| \leqslant M(r) \right\}.$$

In general the classes $S(m,M)$ are not empty. For many classes of normalized holomorphic functions the bounds on modulus of functions are known. Thus if we put $m(r)$ as a lower bound of $|F(z)|$ and $M(r)$ as an upper bound of $|F(z)|$ then a class $S(m,M)$ is a typical example and obviously $F \in S(m,M)$. The classes $S(m,M)$ contain usually some non-univalent functions.

Now we are going to prove a result which gives a solution of a mentioned problem in case of the class $S(m,M)$.

THEOREM 1. Let n be a fixed natural number greater than 1, and let $f \in N_n$, $F \in S(m,M)$. If $f \ll F$ in K_1 then for every $R \in (0,1)$ the inclusion

$$f(K_{r(R)}) \subset F(K_R)$$

holds, where

(2) $$r(R) = r(R;n,S(m,M)) = \sup\{ r \in (0,1) : r^{n-1} M(r) < m(R) \}$$

does not depend on choosing the pair of functions f, F, but only on the classes over which these functions range.

REMARK 1. Theorem 1 is the best possible, that is we can not replace the function $r(R)$ by a bigger function if there exists a univalent function F_e such that for every $r_1, r_2 \in (0,1)$ there exist complex numbers z_1, z_2, $|z_1| = r_1$, $|z_2| = r_2$ for which

$$(3) \qquad |F_e(z_1)/F_e(z_2)| = m(r_1)/M(r_2) .$$

REMARK 2. Theorem 1 gives a possibility to obtain an explicit solution of converse of so called generalized Biernacki problem (see for example [3], [4], [5]) for many classes of analytic functions. It is enough to include the given class in some special class $S(m,M)$. If the extremal function F_e belongs to the given class then the result is best possible.

P r o o f of Theorem 1. The facts $f \prec F$ in K_1, $f \in N_n$, $F \in S(m,M)$ imply that there exists a function $\phi \in B_n$ which satisfies the identity

$$(4) \qquad f(z) \equiv \phi(z)F(z) \qquad \text{for} \quad z \in K_1 .$$

Using the generalized Schwarz's lemma (cf.e.g. [2], p.361) to a function $\Phi \in B_n$ we obtain

$$(5) \qquad |\phi(z)| \leq |z|^{n-1} \qquad \text{for} \quad z \in K_1 .$$

Thus from (4) and (5) we have for $|z| < r < 1$

$$(6) \qquad |f(z)| \leq |z|^{n-1} \max_{|\zeta| \leq |z|} |F(\zeta)| < r^{n-1}M(r) .$$

It means that

$$(7) \qquad f(K_r) \subset \{w : |w| < r^{n-1}M(r)\}.$$

On the other hand, if $F \in S(m,M)$ then for $|z| = R < 1$ we have

$$(8) \qquad |F(z)| \geq m(R) .$$

The function F is holomorphic in K_1 and therefore by (8) we have

$$F(K_R) \subset \{w : |w| < m(R)\}. \tag{9}$$

Now, from (7) and (9) we see that

$$f(K_r) \subset F(K_R)$$

if only r and R satisfy the condition

$$r^{n-1}M(r) < m(R).$$

In particular

$$f(K_{r(R)}) \subset F(K_R)$$

and the proof of Theorem 1 is complete.

P r o o f of Remark 1. If there exists an extremal function F_e in the class $S(m,M)$ which satisfies (3) then the pair of functions

$$f_e(z) = \eta z^{n-1}F_e(z), \qquad F_e(z)$$

with suitably choosen η ($|\eta| = 1$) is an extremal pair. We can choose a complex number η such that

$$f_e(z_1) = F_e(z_2).$$

In this case we put

$$\eta = \frac{F_e(z_1)}{z_2^{n-1}F_e(z_2)} \quad \text{where} \quad |z_1| = r_1 = r, \quad |z_2| = r_2 = r(R).$$

Thus

$$|\eta| = \frac{F_e(z_1)}{|z_2|^{n-1}|F_e(z_2)|} = \frac{m(R)}{[r(R)]^{n-1}M(r(R))} = 1$$

It means that the point $F_e(z_1)$ which is a boundary point of $F(K_R)$ is also a boundary (or interior) point of the domain $f_e(K_{r(R)})$. Therefore no number $\rho > r(R)$ does exist such that

$$f_e(K_\rho) \subset F_e(K_R).$$

It proves that Theorem 1 is best possible.

Now, we use Theorem 1 to solve the converse of the generalized Biernacki problem for the class

$$S^*_\alpha = \left\{ F \in N : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad \text{for} \quad z \in K_1 \right\}, \qquad \alpha \in \langle 0,1) .$$

It is known (cf.e.g. [1]) that if $F \in S^*_\alpha$ and $|z| = r < 1$ then

$$(10) \qquad \frac{r}{(1+r)^{2(1-\alpha)}} \le |F(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}} .$$

If we put

$$(11) \qquad m(r) = \frac{r}{(1+r)^{2(1-\alpha)}}, \quad M(r) = \frac{r}{(1-r)^{2(1-\alpha)}}$$

then $S(m,M) \supset S^*_\alpha$. The function

$$F_e(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$$

belongs to S^*_α and satisfies (3) with $z_1 = r_1$ and $z_2 = -r_2$. Thus from Theorem 1 we have immadiately

COROLLARY 1. Let n be a natural number greater then 1 and let $f \in N_n$, $F \in S^*_\alpha$. If $f \ll F$ in K_1 then for every $R \in (0,1)$ the inclusion $f(K_{r(R)}) \subset F(K_R)$ holds; $r(R) = r(R; n, \alpha)$ is the unique root of the equation

$$r + r^{\frac{n}{2(1-\alpha)}} R^{\frac{1}{2(1-\alpha)}} (1 + R) - 1 = 0$$

which lies in the interval $(0,1)$. The result is best possible.

REMARK 3. For $n = 2$ and $\alpha = 0$ or $\alpha = \frac{1}{2}$ we have

$$r(R;2,0) = \frac{R}{1 + \sqrt{R} + R}$$

$$r(R;2, \tfrac{1}{2}) = \frac{2\sqrt{R}}{\sqrt{4 + 5R} + \sqrt{R}}$$

The proof of Theorem 1 suggests the following generalization.

THEOREM 2. Let n be a fixed natural number greater then 1 and let $f \in N_n$, $F \in S(m,M)$. If $f \ll F$ in K_1 then for every $R \in (0,1)$ and every $G \in S(m,M)$ the inclusion

$$f(K_{r(R)}) \subset G(K_R)$$

holds, where

$$r(R) = r(R;n,S(m,M)) = \sup\{r \in (0,1) : r^{n-1}M(r) < m(R)\}$$

is the same as in Theorem 1. If there exists an extremal function F_e in the class $S(m,M)$ which satisfies (3) then the result is best possible.

P r o o f. Analogously as in the proof of Theorem 1 we obtain the inclusion (7). On the other hand, if $G \in S(m,M)$ then $|G(z)| \geq m(|z|)$ and therefore

$$G(K_R) \supset \{w : |w| < m(R)\}. \tag{12}$$

Now, if r and R satisfy the inequality $r^{n-1}M(r) < m(R)$

then by (7) and (12) we have

$$f(K_r) \subset G(K_R).$$

This proves Theorem 2.

By Remark 1, the result is best possible because we can take $G = F$.

In an analogous way we can generalize Theorem 1 in the paper [6]:

THEOREM 3. Let n be a fixed natural number greater then 1 and let $f \in N_n$, $F \in S(m,M)$. If $f \rightarrow F$ in K_1 then for $|z| = r < 1$ and for every function $G \in S(m,M)$ the following inequality

$$|f(z)| \leqslant T(r)|G(z)|$$

holds, where

$$T(r) = T(r;n,S(m,M)) = M(r^n)/m(r) \ . \tag{13}$$

P r o o f. By our assumptions there exists a function $\omega(z) \in \Omega_n$ such that $f(z) = F(\omega(z))$ for $z \in K_1$. Thus for $|z| = r < 1$ we have (by generalized Schwarz's lemma (cf.e.g. [2], p. 361):

$$|f(z)| = |F(\omega(z))| \leqslant M(|\omega(z)|) \leqslant M(r^n) \ .$$

Now, by (12) we have

$$|f(z)| \leqslant M(r^n) \cdot 1 \leqslant M(r^n)\frac{|G(z)|}{m(r)} = T(r)|G(z)|$$

and the proof is complete.

REMARK 4. Theorem 3 is best possible that is the function $T(r)$ given by (13) cannot be replaced by any smaller function if there exists an extremal function $F_e \in S(m,M)$ such that for any numbers $r_1, r_2 \in (0,1)$ there exist two complex numbers z_1, z_2, $|z_1| = r_1$, $|z_2| = r_2$ such that

(14) $$|F_e(z_1)| = m(r_1), \quad |F_e(z_2)| = M(r_2) .$$

P r o o f. If we put

$$F(z) = G(z) = e^{-i\theta} F_e(ze^{i\theta}), \quad f(z) = e^{-i\theta} F_e(z^n e^{i\theta})$$

then f, F satisfy the hypothesis of Theorem 3. We may choose θ, z_0, $|z_0| = r$ such that the following two conditions

$$e^{i\theta} z_0 = z_1 = re^{i\alpha}$$
$$e^{i\theta} z_0^n = z_2 = r^n e^{i\beta}$$

are satisfied. In particular we can put

$$\theta = (n\alpha - \beta)/(n-1)$$
$$z_0 = r \exp(\alpha - \beta)/(n-1) .$$

Then, by (14) we have

(15) $$|f(z_0)| = |e^{-i\theta} F_e(e^{i\theta} z_0^n)| = |F_e(z_2)| = M(r^n) ,$$

(16) $$|G(z_0)| = |e^{-i\theta} F_e(e^{i\theta} z_0)| = |F_e(z_1)| = m(r) .$$

Now, (15) and (16) imply the equality in (13) and therefore the function $T(r)$ can not be replaced by any smaller function.

We can also obtain the two following results:

THEOREM 4. Let n be any fixed, natural number, greater then 1 and let $f \in N_n$, $F \in S(m,M)$. If $f \dashv F$ in K_1, then for every $R \in (0,1)$ and for every function $G \in S(m,M)$ the following inclusion

$$(17) \qquad f(K_{r(R)}) \subset G(K_R)$$

holds, where

$$(18) \qquad r(R) = \sqrt[n]{M^{-1}(m(R))}$$

is the smallest positive root of the equation

$$M(r^n) = m(R) .$$

P r o o f. By our assumptions we have $f(z) = F(\omega(z))$ where $\omega \in \Omega_n$ and therefore

$$(19) \quad f(K_r) \subset \{w : |w| < \sup_{\substack{|\zeta| \le r^n \\ F \in S(m,M)}} |F(\zeta)|\} \subset \{w : |w| < M(r^n)\}.$$

Thus from (12) i (19) we have

$$f(K_r) \subset G(K_R)$$

if only r, R satisfy the inequality

$$M(r^n) < m(R) .$$

Therefore if $r(R)$ is given by (18) then (17) holds and the theorem is proved.

THEOREM 5. Let n be a fixed natural number greater then 1 and let $f \in N_n$, $F \in S(m,M)$. If $f \ll F$ in K_1, then for every function $G \in S(m,M)$ and for every z, $|z| = r < 1$

the inequality

$$|f(z)| \leqslant T_1(r)|G(z)|$$

holds, where $T_1(r) = T_1(r;n,S(m,M))$ is given by the formula

$$T_1(r) = r^{n-1}\frac{M(r)}{m(r)} . \tag{20}$$

P r o o f. By our assumptions we have $f(z) = \Phi(z)F(z)$ where $\Phi \in B_n$. Therefore if $|z| = r<1$ then

$$|f(z)| = |\Phi(z)||F(z)| \leqslant |z|^{n-1}M(|z|) \leqslant r^{n-1}M(r)\frac{|G(z)|}{m(r)} =$$

$$= T(r)|G(z)|$$

where $T(r)$ is given by (20). Thus our theorem is proved.

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STRESZCZENIE

W pracy badane są zależności między podporządkowaniem a inkluzją obrazów kół koncentrycznych w przypadku gdy $f \in N_n$, $n \geqslant 2$, a $F \in S(m,M)$.

Резюме

В работе исследовано зависимость между подчинением а включением образов концентрических кругов в случае когда $f \in N_n$ $n \geqslant 2$, а $F \in S(m,M)$.