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On the Radius of Univalence for the Integral of $f'(z)^{\alpha}$

Promień jednolistności całki $f'(z)^{\alpha}$

Радиус однолистности интеграла $f'(z)^{\alpha}$

In this note we deal with some classes of holomorphic functions f in the unit disc $K = \{z : |z| < 1\}$ which have the form

$$(1) \qquad f(z) = z + a_2 z^2 + \dots , \qquad z \in K.$$

By S we denote the family of holomorphic and univalent functions in K which have the form (1) and by $LV(\beta, k)$ the class of β - close - to - V_k functions [1], [7].

Let us consider the following integral operator

$$(2) \qquad F(z) = J_{\alpha,\beta}[f](z) = \int_0^z [f'(t)]^{\alpha} \left[\frac{f(t)}{t}\right]^{\beta} dt ,$$

where α and β are arbitrary fixed real numbers. The powers are defined via the branch of log for which $\log F'(0)=0$.

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Many authors e.g. [3], [4], [6] considered the behavior of (2) in the class S as well as in some subclasses of S. Namely, they obtained the results concerning the bounds for α and β (also in the complex case) which preserve univalence of F or other properties.

In this note we find the bound from below for the radius of univalence $r_u(S)$ of the integral

$$(3)\qquad F(z) = J_\alpha[f](z) = \int_0^z [f'(t)]^\alpha dt, \quad f \in S(LV(\beta,k)).$$

It is known that the integral (3) is univalent for $|\alpha| \leq \frac{1}{4}$ and is not always univalent for $|\alpha| > \frac{1}{3}$, $\alpha \neq 1$ (Pfaltzgraff [4], Royster [6]). On the other hand in [7] it was shown that the integral (3) is univalent for all $\alpha \in [0,1]$ in the disc $|z| < r_u(S)$, where $r_u(S) > 0.81$.

Here we improve the last result giving also the dependence of $r_u(S)$ $(r_u(LV(\beta,k)))$ on α. We have the following

THEOREM 1. Let $f \in S$. Then the function (3) is univalent for $|z| < r_u(S)$, where $r_u(S) \geq r_\alpha$ and r_α is the root of the equation

$$(4)\quad \begin{cases} |\alpha|\left(\frac{\pi}{2} + \log\frac{2r}{1-r^2}\right) + 2|1-\alpha| \operatorname{arc\,tg} r = \pi, & \text{if } \alpha \in [-1, \frac{5}{3}] \\ r = \operatorname{tg}\dfrac{\pi}{2(2|\alpha| + |1-\alpha|)}, & \text{if } \alpha \in (-\infty,-1) \cup (\frac{5}{3}, +\infty). \end{cases}$$

Moreover we have $r_{1/4} > 0.998$, $r_{1/3} > 0.991$.

We begin with the following lemma which has an independent interest. This lemma gives the extension of the family $J_{\alpha,\beta}(H)$ up to the linear - invariant family $\hat{H}$ for an arbitrary linear - invariant family H in the sense of Pommerenke [5].

LEMMA. *Let H be a linear - invariant family of holomorphic functions f of the form* (1) *and let $J_{\alpha,\beta}(H)$ denotes the set of the functions F given by* (2). *Then the family $\hat{H}$ of functions G given by the formula*

$$(5) \qquad G(z) = \hat{J}_{\alpha,\beta}[f](z) = \left[\frac{\xi}{f(\xi)}\right]^{\beta} \int_0^z \frac{[f'(s)]^{\beta}[f(s) - f(\xi)]^{\beta}}{(s-\xi)^{\beta}(1-\bar{\xi}s)^{2-2\alpha-\beta}}\, ds$$

where $f \in H$ and ξ is an arbitrary point in K, is the minimal linear - invariant family containing the family $J_{\alpha,\beta}(H)$.

Proof. It is known that if $f \in H$ then for every $\xi \in K$ also the function

$$(6) \qquad g(z) = \Lambda_{\xi}[f](z) = \frac{f\left(\frac{z+\xi}{1+\bar{\xi}z}\right) - f(\xi)}{(1-|\xi|^2)f'(\xi)} \in H .$$

Moreover, we have $\Lambda_{\xi}(H) = H, \quad \xi \in K$.

Let us observe that the set of functions

$$(7) \qquad \hat{H} = \bigcup_{\xi \in K} \Lambda_{\xi}[J_{\alpha,\beta}(H)] = \bigcup_{\xi \in K} \Lambda_{\xi}\{J_{\alpha,\beta}[\Lambda_{-\xi}(H)]\}$$

has the properties mentioned in the Lemma. In fact $\hat{H}$ is the linear - invariant family containing $J_{\alpha,\beta}(H)$. The minimal property of H follows from the formula (6) because every other linear - invariant family containing $J_{\alpha,\beta}(H)$ must contain $\hat{H}$.

Now in order to get the formula (5) we use the relation (7). For $g = \Lambda_{\xi}(J_{\alpha,\beta}[f])$ we have

$$g(z) = k\left[J_{\alpha,\beta}[f]\left(\frac{z+\xi}{1+\bar{\xi}z}\right) - J_{\alpha,\beta}[f](\xi)\right] =$$

$$= k\int_{\xi}^{\frac{z+\xi}{1+\bar{\xi}z}} [f'(t)]^{\alpha}\left[\frac{f(t)}{t}\right]^{\beta} dt,$$

where k is a constant depending on ξ.

Changing the variable in the above formula by $t = \frac{s+\xi}{1+\bar{\xi}s}$, we get

$$g(z) = k_1 \int_0^z \left[f'\left(\frac{s+\xi}{1+\bar{\xi}s}\right)\right]^{\alpha}\left[\frac{f\left(\frac{s+\xi}{1+\bar{\xi}s}\right)}{\frac{s+\xi}{1+\bar{\xi}s}}\right]^{\beta} \frac{ds}{(1+\bar{\xi}s)^2},$$

k_1 is a constant.
Putting into above formula the function $\Lambda_{-\xi}[f]$ instead of f after simplifications we obtain (5) which ends the proof of the Lemma.

P r o o f. According to the Lemma for $\beta = 0$ the family $\hat{H} = \hat{S}$ for $H = S$ consists of the functions F given by

$$\hat{F}(z) = \int_0^z \frac{[f'(t)]^{\alpha}}{(1-\xi t)^{2-2\alpha}}\, dt, \qquad f\in S, \quad \xi\in K. \tag{8}$$

So far we know that $\hat{S}$ is the linear - invariant family we have the following formula for the radius of univalency $r_u(\hat{S}) = r_u$ [5]:

$$r_u = \frac{r_0}{1+\sqrt{1-r_0^2}}, \tag{9}$$

where r_0 is the radius of the largest disc with centre in the origin in which every function $f\in\hat{S}$ is different from zero except for the origin. The reasoning as in [5, Satz. 2.6] implies that if $f \in \hat{S}$ and $f(r_0) = 0$, then $\arg f'(r_0) = \pm 2\pi$

Now we see that in order to get r_u we should find r_0.

Using the well known exact estimate for $|\arg f'(z)|$, $f \in S$ [2], we have from (8) ($|z| = r$):

$$(10) \quad |\arg \hat{F}'(z)| \leq |\alpha||\arg f'(z)| + 2|1-\alpha| \arg(1-\bar{\xi}z) \leq$$

$$\begin{cases} |\alpha|\left(\pi + \log \dfrac{r^2}{1-r^2}\right) + 2|1-\alpha| \arc\sin r & \text{if } -1 \leq \alpha \leq \frac{5}{3} \\ 4|\alpha| \arc\sin r + 2|1-\alpha| \arc\sin r & \text{if } \alpha < -1 \text{ or } \alpha > \frac{5}{3}. \end{cases}$$

From (10) and the above remarks follows that $r_0 = r_0(\hat{S}) \leq$ $\leq r_\alpha^0$ where r_α^0 is the unique root of the equations

$$(11) \quad \begin{cases} |\alpha|\left(\pi + \log \dfrac{r^2}{1-r^2}\right) + 2|1-\alpha| \arc\sin r = 2\pi & \text{if } -1 \leq \alpha \leq \frac{5}{3} \\ \arc\sin r = \dfrac{\pi}{2|\alpha| + |1-\alpha|} & \text{if } \alpha < -1 \text{ or } \alpha > \frac{5}{3}. \end{cases}$$

The formulae (9) and (11) lead to the result (4), which ends the proof of Theorem 1.

Analogous results may be obtained for other classes of holomorphic functions for which the bound of $\arg f'(z)$ is known.

We get such result for the quite wide class $LV(\beta, k)$ [1], [7].

THEOREM 2. *Let* $f \in LV(\beta, k)$, $\beta \geq 0$, $k \geq 2$. *Then the function* (3) *is univalent for* $|z| < r_u(LV(\beta, k))$ *where* $r_u \geq r_\alpha'$ *and*

(12) $$r' = \min\left\{1, \operatorname{tg}\frac{\pi}{|\alpha|(2\beta + k) + 2|1-\alpha|}\right\},$$ is real

The proof of the Theorem 2 can be established in the same way as Theorem 1. It is only necessary to take into account that $|\arg f'(z)| \leq (2\beta + k)\arcsin|z|$ for $f \in LV(\beta, k)$ [7].

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STRESZCZENIE

Niech S oznacza klasę funkcji f, $f(0) = 0$, $f'(0) = 1$, holomorficznych i jednolistnych w kole $K = \{z : |z| < 1\}$.

W pracy podaje się oszacowanie od dołu dla promienia jednolistności $r_u(S)$ całki

$$F(z) = \int_0^z f'(t)^{\alpha} dt, \qquad f \in S, \qquad \alpha \in \mathbb{R}$$

Mianowicie dowodzi się, że $r_u(S) \geq r_{\alpha}$, gdzie r_{α} dane jest jako pierwiastek równań (4).

Analogiczny problem został przedstawiony dla klasy $LV(\beta, k)$ [7].

Резюме

Пусть S-класс функций $f, f(0)=0, f'(0)=1$, голоморфных и однолистных в круге $K=\{z: |z|<1\}$. В настоящей работе получена оценка снизу для радиуса однолистности $r_u(S)$ интеграла

$$F(z)=\int_0^z f'(t)^{\alpha} dt, \quad f \in S, \quad \alpha \in R$$

Показано, что $r_u(S) \geq r_{\alpha}$, где r_{α} является корнем уравнения (4). Аналогичная задача решена для класса $LV(\beta,k)$[7].