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On the Non-existence of Parabolical Podkovyrin Quasi-connections

O nieistnieniu parabolicznych quasi-koneksji Podkowyrina

О несуществовании параболических квази-связности Подковырина

A.S. Podkovyrin in [4,3,6] and other authors (e.g. [7]) have investigated structures on 2n-dimensional manifold M provided with tensors a, g, E where a is a covector, g is a symmetrical nondegenerate (0,2) tensor, and E is (1,1) tensor in the form

$$(1) \qquad E = \left[\begin{array}{c|c} 0 & e \\ \hline \varepsilon e & 0 \end{array}\right] \qquad \det e \neq 0$$

and such that $E \cdot E = \varepsilon I$, where ε is either 1 (hyperbolical case), or -1 (elliptical case), or 0 (parabolical case).

A connection ∇ is said to be Podkovyrin connection if for arbitrary vector fields v, u, w the following conditions hold

(2) $$\nabla E = 0$$

(3) $$\nabla_v g(u,w) = a(v)\cdot g(u,E(w)).$$

Our task is to consider the parabolic case i.e. $\varepsilon = 0$. The structure determined by the tensor E such that $E^2 = 0$, rank $E = n$, is usually called an almost tangent structure [1]. Now on that occasion we shall also investigate all quasi--connections determined by (2) on its almost tangent structure.

The pair (C^i_j, Φ^i_{jk}) where C^i_j is a (1,1) tensor and Φ^i_{jk} is a set of functions for which the transformation rule is as follows

(4) $$\Phi^a_{jk}A^{i'}_a = C^a_j A^{i'}_{ak} + A^{a'}_j A^{b'}_k \Phi^{i'}_{a'b'}$$

is said to be quasi-connection on the manifold M. A covariant derivation ∇ with respect to the pair (C^i_j, Φ^i_{jk}) is in the form

(5) $$\begin{aligned} \nabla_t v^i &= C^a_t \partial_a v^i + v^a \Phi^i_{ta} \\ \nabla_t w_i &= C^a_t \partial_a w_i - \Phi^a_{ti} w_a \\ \nabla_z z^i_j &= C^a_t \partial_a z^i_j + z^a_j \Phi^i_{ta} - \Phi^a_{tj} z^i_a \\ \nabla_t g_{ij} &= C^a_t \partial_a g_{ij} - \Phi^a_{it} g_{aj} - \Phi^a_{jt} g_{ia} \end{aligned}$$

Y.-C. Wong in [9] has given reasons for the investigation of quasi-connection as well as another definition and general theory of this one. If C^i_j is nondegenerate tensor then $\Gamma^i_{jk} := \overset{-1}{C}{}^t_j \Phi^i_{tk}$ are classical coefficients of a connection,

as one can straightforward check.
For our purpose is necessary to recollect certain theorem concerning generalized inverse of matrices and some its generalization. Theory of generalized inverse of matrices was developed in statistics mostly in the theory of linear models.

To begin with we remind the following theorem:

THEOREM 1 (C.R. Rao, S.K. Mitra [7]). If A is an arbitrary $n \times n$ matrix and A^- is any matrix satisfying the relation $AA^-A = A$, then a necessary and sufficient condition for the existence of the solution of equation

$$Ax = y \tag{6}$$

is that $AA^-y = y$. If this holds then all solutions have the form

$$x = A^-y + (I - A^-A)w \tag{7}$$

where w is an arbitrary vector.

COROLLARY 2 (theorem of M. Obata [2],[3],[8]). If A is projection operator i.e. $A \circ A = A$ then $A^* := I - A$ is such that $A^* \circ A^* = A^*$ and $A^* \circ A = A \circ A^* = 0$. It is easy to see that in this case we choose $A^- = I$ and that the condition $AA^-y = y$ reduces to $A^*y = 0$. All solutions of the equation (6) have the form

$$x = y + A^*w \tag{8}$$

where w is arbitrary.
We are going to show slight generalization of the above Theorem 1.

THEOREM 3. *If A and B are arbitrary $n\times n$ matrices and A^-, B^- are such that $AA^-A = A$ and $BB^-B = B$ then necessary and sufficient conditions for the existence of solutions of system of equations*

$$(9)\qquad \begin{aligned} Ax &= y \\ Bx &= z \end{aligned}$$

are

$$(10)\qquad \begin{aligned} AA^-y = y, \quad BB^-z = z, \quad AB^-BA^-A = AB^-B \\ AB^-BA^-y = AB^-z\ . \end{aligned}$$

At that time all solutions have the form

$$(11)\qquad x = A^-y + B^-z - B^-BA^-y + (I - B^-B)(I - A^-A)w$$

where w is an arbitrary vector.

COROLLARY 4 (Lemma of Cz. Tokarczyk [8]). If A and B are projection operators then $A^- = B^- = I$ and conditions for the existence of the solution are

$$(12)\qquad \begin{aligned} A^*y = 0, \quad B^*z = 0 \\ A\circ B\circ A = A\circ B, \qquad A\circ By = Az \end{aligned}$$

and all solutions are given in the form

$$(13)\qquad x = y + z - By + B^*\circ A^*w\ .$$

P r o o f of theorem. We are going to apply the Theorem 1 to the following equation

$$\begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} y \\ z \end{bmatrix} \tag{14}$$

$\Leftarrow$) if (10) holds then the matrix $\left[(I - B^-B)A^-, B^-\right]$ is a generalized inverse of $\begin{bmatrix} A \\ B \end{bmatrix}$.

In fact, we have

$$\begin{bmatrix} A \\ B \end{bmatrix} \left[(I - B^-B)A^-, B^-\right] \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix} A \\ B \end{bmatrix} \left[(I - B^-B)A^-, B^-\right] \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} \tag{16}$$

In view of the Theorem 1 the solution exists and it may be written in the following form

$$x = \left[(I - B^-B)A^-, B^-\right] \begin{bmatrix} y \\ z \end{bmatrix} + \left(I - \left[(I - B^-B)A^-, B^-\right] \begin{bmatrix} A \\ B \end{bmatrix}\right) w \tag{17}$$

or

$$x = A^-y + B^-z - B^-BA^-y + (I - B^-B)(I - A^-A)w \tag{18}$$

$\Rightarrow$) one can easy check by substituting (18) into equations (9).

Now, we can turn back to our problem. Let's consider the equation (2) only. In local form

$$(\delta^k_s E^r_p - E^k_s \delta^r_p)\phi^p_{kq} = E^k_q \partial_k E^r_s \tag{19}$$

Let us also consider $\phi_q := \{\phi^p_{kq}\}_{(p,k)}$ as $4n^2$ - tuple ordered in a lexicographic manner. In this moment we can write (19) as

$$(E \otimes I - I \otimes E^t)\,\Phi_q = T_q$$

(20)

$$F \cdot \Phi_q = T_q$$

where T_q is $\{E^k_q \partial_k E^r_s\}_{(r,s)}$ ordered in similar way as Φ_q and E^t denotes a transposition of E. Let us write the matrix F in a box form

$$\text{(21)} \qquad F = \left[\begin{array}{c|c} -I_n \otimes E^t & e \otimes I_{2n} \\ \hline 0 & -I_n \otimes E^t \end{array}\right]$$

where I_k denotes the identity $k \times k$ matrix .

LEMMA 5. *If* F *is above mentioned matrix, then matrix* F^-

$$\text{(22)} \qquad F^- = \left[\begin{array}{c|c} 0 & I_{2n^2} \\ \hline e^{-1} \otimes I_{2n} & -e^{-1} \otimes E^t \end{array}\right]$$

is such that $FF^-F = F$ *and* $FF^-T_k = T_k$.

P r o o f.

$$FF^- = \left[\begin{array}{c|c} -I_n \otimes E^t & e \otimes I_{2n} \\ \hline 0 & -I_n \otimes E^t \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & I_{2n^2} \\ \hline e^{-1} \otimes I_{2n} & -e^{-1} \otimes E^r \end{array}\right]$$

(23)

$$= \left[\begin{array}{c|c} I_{2n^2} & -2I_n \otimes E^t \\ \hline -e^{-1} \otimes E^t & 0 \end{array}\right]$$

$$FF^{-}F = \left[\begin{array}{c|c} I_{2n^2} & 2I_n \otimes E^t \\ \hline -e^{-1} \otimes E^t & 0 \end{array}\right] \left[\begin{array}{c|c} -I_n \otimes E^t & e \otimes I_{2n} \\ \hline 0 & -I_n \otimes E^t \end{array}\right] =$$

(23)

$$= \left[\begin{array}{c|c} -I_n \otimes E^t & e \otimes I_{2n} \\ \hline 0 & -I_n \otimes E^t \end{array}\right] = F$$

It is not difficult to see that from the form of $E = \left[\begin{array}{c|c} 0 & e \\ \hline 0 & 0 \end{array}\right]$ we can write T_k in the shape

$$(24) \qquad T_k = \left[{}_n0,\ \varepsilon_{k1},{}_n0,\ \varepsilon_{k2},{}_n0,\dots,{}_n0,\ \varepsilon_{kn},{}_n0,\dots,{}_n0\right] =: $$
$$=: \left[T^1_{k},{}_{2n^2}0\right]$$

where ${}_n0$ is n-dimensional zero, ε_{ki} consists of n adequate elements. The term T^1_k consists of $2n^2$ elements. On account of the form of FF^{-} it is sufficient to show that

$$(25) \qquad -(e^{-1} \otimes E^t)T^1_k = 0$$

The left-hand member of (25) has the form

$$(26) \qquad \left[\begin{array}{cccc} 0_n, & 0_n,\dots, & 0_n, & 0_n \\ -e^{-1}E^t, & 0_n,\dots, & -e^{-1}E^t, & 0_n \\ 0_n, & 0_n,\dots, & 0_n, & 0_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0_n, & 0_n,\dots, & 0_n, & 0_n \\ -e^{-1}E^t, & 0_n,\dots, & -e^{-1}E^t, & 0_n \end{array}\right] \left[\begin{array}{c} {}_n0 \\ \varepsilon_{k1} \\ {}_n0 \\ \\ \\ {}_n0 \\ \varepsilon_{kn} \end{array}\right]$$

(O_n denotes the zero $n \times n$ matrix) it is evident that this product is equal to zero.

By virtue of above lemma and of the Theorem 1 we can state the following

COROLLARY 6. The coefficients Φ^i_{jk} of all structure quasi-connections (E^i_j, Φ^i_{jk}) of an almost tangent structure E of 2n-dimensional manifold are

$$\Phi_k = F^- T_k + (I - F^- F)W \tag{27}$$

(cf. [1]).

Let us consider equation (3) in local form

$$(\delta^k_r g_{ps} + g_{rp}\delta^k_s)\Phi^p_{kq} = E^t_q \partial_t g_{rs} - a_q b_{wr} E^w_s \tag{28}$$

or after a contraction with g^{sz}

$$\frac{1}{2}(\delta^k_r \delta^z_p + g_{rp} g^{kz})\Phi^p_{kq} = \frac{1}{2}(E^t_q g^{sz}\partial_t g_{rs} - a_q g_{wr} E^w_s g^{sz}) \tag{29}$$

or as a matrix equation

$$\begin{gathered} \frac{1}{2}(I \otimes I + g \otimes g^{-1})\Phi_q = K_q \\ \Omega \cdot \Phi_q = K_q \end{gathered} \tag{30}$$

where

$$\begin{aligned} \Omega &= \frac{1}{2}(I \otimes I + g \otimes g^{-1}) \\ \Phi_q &= \{\Phi^p_{kq}\}_{(p,k)} \\ K_q &= \left\{\frac{1}{2}(g^{sz} E^t_q \partial_t g_{rs} - a_q g_{wr} E^w_s g^{sz})\right\}_{(z,r)} \end{aligned} \tag{31}$$

are ordered like in (19).

It is easy to check that $\Omega \circ \Omega = \Omega$ hence Ω is a well-known Obata operator. It is also clear that $\Omega^{-} = I$. Because (0,2) tensor g is symmetrical and non-degenerate we can represent g in the box form

$$g = \left[\begin{array}{c|c} g_1 & g_2 \\ \hline g_2^t & g_3 \end{array}\right] \tag{32}$$

as well

$$\Omega = \left[\begin{array}{c|c} I_{2n^2} + g_1 \otimes g^{-1} & g_2 \otimes g^{-1} \\ \hline g_2^t \otimes g^{-1} & I_{2n^2} + g_3 \otimes g^{-1} \end{array}\right] \tag{33}$$

THEOREM 7. There are no quasi connections ∇ such that both (2), (3) hold simultaneously.

P r o o f. We shall show that condition $F\Omega F^{-} F = F\Omega$ (cf. Theorem 3) holds iff the tensor g has the form

$$g = \left[\begin{array}{c|c} g_1 & 0 \\ \hline 0 & e^{-1} g_1 e \end{array}\right] \tag{34}$$

Then it will be a contradiction because the right-hand member of (3) with the tensor (34) cannot be symmetrical but the left-hand member of (3) is symmetrical by the definition.

Let us consider

$$F\Omega = \left[\begin{array}{c|c} -I_n \otimes E^t & e \otimes I_{2n} \\ \hline 0 & -I_n \otimes E^t \end{array}\right]\left[\begin{array}{c|c} I_{2n^2} + g_1 \otimes g^{-1} & g_2 \otimes g^{-1} \\ \hline g_2^t \otimes g^{-1} & I_{2n^2} + g_3 \otimes g^{-1} \end{array}\right] \tag{35}$$

$$= \left[\begin{array}{c|c} -I_n \otimes E^t - g_1 \otimes E^t g^{-1} + e g_2^t \otimes g^{-1} & -g_2 \otimes E^t g^{-1} + e \otimes I_{2n} + e g_3 \otimes g^{-1} \\ \hline -g_2^t \otimes E^t g^{-1} & -I_n \otimes E^t - g_3 \otimes E^t g^{-1} \end{array}\right]$$

(36) $$F^{-}F = \left[\begin{array}{c|c} 0 & -I_n \otimes E^t \\ \hline -e^{-1} \otimes E^t & I_{2n^2} \end{array}\right]$$

Taking into considerations (35) and (36) we have

(37) $$F\Omega F^{-}F =$$

$$= \left[\begin{array}{c|c} g_2 e^{-1} \otimes E^t g^{-1} E^t - I_n \otimes E^t & g_1 \otimes E^t g^{-1} E^t - g_2 \otimes E^t g^{-1} + e \otimes I_{2n} \\ - e g_3 e^{-1} \otimes g^{-1} E^t & - e g_2^t \otimes g^{-1} E^t + e g_3 \otimes g^{-1} \\ \hline g_3 e^{-1} \otimes E^t g^{-1} E^t & g_2^t \otimes E^t g^{-1} E^t - I_n \otimes E^t - \\ & - g_3 \otimes E^t g^{-1} \end{array}\right]$$

And this matrix should be equal to (35), so we have the following identies

(38) $$- g_1 \otimes E^t g^{-1} + e g_2^t \otimes g^{-1} = g_2 e^{-1} \otimes E^t g^{-1} E^t - e g_3 e^{-1} \otimes g^{-1} E^t$$

(39) $$g_3 e^{-1} \otimes E^t g^{-1} E^t = - g_2^t \otimes g^{-1} E^t$$

(40) $$g_1 \otimes E^t g^{-1} E^t = e g_2^t \otimes g^{-1} E^t$$

(41) $$g_2^t \otimes E^t g^{-1} E^t = 0$$

Because of (41) we have three possibilities

(I) $g_2^t = 0$ and $E^t g^{-1} E^t \neq 0.$

From (39) we have $g_3e^{-1}\otimes E^tg^{-1}E^t = 0$ hence $g_3 = 0$. And from (38) we have $g_1\otimes E^tg^{-1} = 0$ as well from (40) we obtain $g_1\otimes E^tg^{-1}E^t = 0$ hence $g_1 = 0$. It is a contradiction because of $g \neq 0$.

(II) $g_2^t \neq 0$ and $E^tg^{-1}E^t = 0$

From (40) we have $eg_2^t\otimes g^{-1}E^t = 0$ and hence $g^{-1}E^t = 0$ as well from (39) we obtain $g_2^t\otimes E^tg^{-1} = 0$ and hence $E^tg^{-1} = 0$. From (38) we see that $eg_2^t\otimes g^{-1} = 0$ and hence $g_2^t = 0$. It is a contradiction.

(III) $g_2^t = 0$ and $E^tg^{-1}E^t = 0$

From (38) we have

(42) $$g_1\otimes E^tg^{-1} = eg_3e^{-1}\otimes g^{-1}E^t$$

It is easy to check that (42) holds iff

(43) $$g = \left[\begin{array}{c|c} g_1 & 0 \\ \hline 0 & e^{-1}g_1e \end{array}\right]$$

and e is orthogonal matrix. This fact finishes the proof.

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STRESZCZENIE

W pracy badano istnienie parabolicznej quasi-koneksji Podkowyrina tj. spełniającej warunki (2), (3) oraz $E \circ E = 0$.

Za pomocą uogólnienia tw. Rao-Mitry udowodniono, że taka quasi-koneksja nie istnieje.

Резюме

В работе исследуется существование параболической квази-связности Подковырина, то есть такой, которая выполняет условия /2/, /3/, а также E o E = 0.

С помощью обобщения теоремы Рао-Митры доказывается несуществование такой квази-связности.