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On a Subclass of Bounded Typically-Real Functions

O pewnej podklasie funkcji typowo-rzeczywistych ograniczonych

О некотором подклассе типично-вещественных органических функций

INTRODUCTION

Let M, $M>1$ be a given number and let z_0, $-1<z_0<1$ be fixed.

Denote by $T_M(z_0)$ the class of functions of the form

(1) $$f(z) = a_1 z + a_2 z^2 + \dots$$

in the unit disk K that have there the following integral representation

(2) $$f(z) = \int_{-1}^{1} G(z,t)\, d\mu(t)$$

where μ is a probability measure on $[-1,1]$ and $G(z,t)$ is given by the formula

$$(3) \qquad G(z,t) =$$

$$= \frac{2z}{2t\frac{z}{M} + k(1-2tz+z^2) + \sqrt{2t\frac{z}{M} + k(1-2tz+z^2) - 4(\frac{z}{M})^2}}$$

and

$$k = k(M,t,z_0) = \frac{M^2 - 2tMz_0 + z_0^2}{M^2(1-2tz_0+z_0^2)}$$

Note that $T_\infty(z_0) = T(z_0)$ is identical to the class introduced by B. Piłat [4] while in the limit case $z_0 \longrightarrow 0$ one obtains the class $T_M(0)$ discussed ealier by Z. Lewandowski and S. Wajler [3].

In this note we prove that functions of the class $T_M(z_0)$ are typically-real in K, (i.e. $\operatorname{Im} z \operatorname{Im} f(z) > 0$, $z \neq \bar{z}$ and $f(z)$ is real for $z = \bar{z}$) satisfy the so called Montel conditions

$$f(0) = 0, \qquad f(z_0) = z_0$$

and are uniformly bounded by M, $|f(z)| < M$.

Moreover, we find the variability region of f within the class $T_M(z_0)$ and sharp estimates for initial Taylor cioefficients of f.

2. MAIN RESULTS

We start with properties of the function $G(z,t)$

LEMMA. The function $G(z,t)$, $t \in [-1,1]$ maps K onto disc $|w| < M$ slit along two segments that lay on the real axis and that include points M, $-M$, respectively.

P r o o f. Put

$$(4)\qquad a = a(z,t) = \frac{Mz(1 - 2tz_0 + z_0^2)}{(M^2 - 2tMz_0 + z_0^2)(1 - 2tz + z^2)}$$

Then

$$(5)\qquad G(z,t) = \frac{2aM}{2at + 1 + \sqrt{(2at + 1)^2 - 4a^2}}$$

Setting $z = e^{i\theta}$ we note that $a(e^{i\theta},t)$ is real. Moreover, if

$$(6)\qquad \left[2a(e^{i\theta},t)t + 1\right]^2 - 4a^2(e^{i\theta},t) \geqslant 0$$

then $G(e^{i\theta},t)$ is real.

If

$$(7)\qquad \left[2a(e^{i\theta},t)t + 1\right]^2 - 4a^2(e^{i\theta},t) < 0$$

then $G(e^{i\theta},t) = M^2$.

It proves the lemma.

Let $\mathfrak{M}$ stand for all probability measures μ on the interval $[-1,1]$.

We have

$$(8)\qquad G'(0,t) = \frac{M^2(1 - 2tz_0 + z_0^2)}{M^2 - 2tMz_0 + z_0^2} > 0, \qquad t \in [-1, 1]$$

Hence, since $G(z,t)$ is univalent and it maps K onto a domain symmetric w.r.t. the real axis we conclude that

$$\operatorname{Im} z \cdot \operatorname{Im} f(z) > 0, \qquad z \in K, \quad z \neq \bar{z}$$

It proves that all functions of the class $T_M(z_0)$ are typically-real.

It easy to see, that $T_M(z_0)$ is a convex and compact family of analytic functions. It follows from a result of Brickman, MacGregor, Wilken [1] that all extreme points f of $T_M(z_0)$ are of the form

$$f(z) = G(z,t), \qquad t \in [-1,1].$$

However, $T_M(z_0)$ is not the convex hull of the class of all typically-real univalent and bounded functions in K. If it were it would imply that coefficients of typically-real univalent functions in K are majorized by corresponding Taylor coefficients of $G(z,t)$. But it would contradict to a result of O. Tammi [5].

THEOREM 1. The variability region of $f(z)$ within the class $T_M(z_0)$ is a convex set bounded by the curve $w(t) = G(z,t)$, $t \in [-1,1]$ and the segment $[G(z,-1), G(z,1)]$.

P r o o f. In view of the well-known result of Carathéodory [2] it is sufficient to determine the convex hull of the curve $w(t) = G(z,t)$. We are going to prove that this curve is a part of a closed convex curve.

For we put

$$(9) \qquad \begin{cases} \zeta = \frac{1}{z} + z \\ l = l(t,M,\zeta) = \frac{2t}{M} + k(\zeta - 2t) \end{cases}$$

and we have

$$(10) \qquad w = G(z,t) = G(\zeta,t) = \frac{2}{l + \sqrt{l^2 - \frac{4}{M^2}}}$$

or

$$w^2 = wlM^2 - M^2 . \tag{11}$$

Let $w = u + iv$, $l = \xi + i\eta$. Thus (11) takes the form

$$\begin{cases} u^2 - v^2 = M^2u\xi - M^2v\eta - M^2 \\ 2uv = M^2u\eta + M^2v\xi \end{cases} \tag{12}$$

By elimination of t one gets ultimately the equation

$$\begin{aligned} & yz_0u^3 + (1 + z_0^2 - xz_0)v^3 + (1 + z_0^2 - xz_0)u^2v + \\ (13)\quad & + yz_0uv^2 - y(M^2 + z_0^2)u^2 - y(M^2 + z_0^2)u^2 - y(M^2 + z_0^2)v^2 + \\ & + M^2yz_0u - M(1 + z_0^2 - xz_0)v = 0 \end{aligned}$$

Noting that $G(\xi,-\infty) = G(\xi,+\infty) = 0$ and taking into consideration (3) we conclude that the equation (13) represents a continuous closed closed curve that passes through the origin.

Now, consider an arbitrary straight line given by

$$\begin{cases} u = u_0 + \alpha t \\ v = v_0 + \beta t \end{cases} \qquad t \in [-\infty, +\infty] \tag{14}$$

The system of equations (13) and (14) has at most 3 solutions for t. If the straight line (14) has more than two points in common with the curve (13) then their number must be even. Since it is impossible, we conclude that any straight line intersects the curve (13) at most at two points. Hence the curve (13) is convex and Theorem 1 follows.

THEOREM 2. Let $f(z) = a_1 z + a_2 z^2 + \ldots \in T_M(z_0)$. Then

$$\frac{M^2(1 - |z_0|)^2}{(M - |z_0|)^2} \leqslant a_1 \leqslant \frac{M^2(1 + |z_0|)^2}{(M + |z_0|)^2},$$

$$- 2M^2(M - 1)(M - z_0^2)\,\frac{(1 + z_0)^2}{(M + z_0)^2} \leqslant$$

$$\leqslant a_2 \leqslant \begin{cases} a_2(t_0), & 0 < t_0 < 1 \\ 2M^2(M - 1)(M - z_0^2)\,\dfrac{(1 - z_0)^2}{(M - z_0)^4}, & t_0 \geqslant 1 \end{cases}$$

for $0 < z_0 < 1$,

$$\left.\begin{array}{ll} a_2(t_0), & -1 < t_0 < 1 \\ - 2M^2(M - 1)(M - z_0^2)\,\dfrac{(1 + z_0)^2}{(M + z_0)^4}, & t_0 \leqslant -1 \end{array}\right\} \leqslant a_2 \leqslant$$

$$\leqslant 2M^2(M - 1)(M - z_0^2)\,\frac{(1 - z_0)^2}{(M - z_0)^4},$$

for $-1 < z_0 < 0$,

where

$$a_2(t) = 2M^2(M - 1)(M - z_0^2)\,\frac{t(1 - 2tz_0 + z_0^2)}{(M^2 - 2tMz_0 + z_0^2)^2},$$

$$t_0 = \frac{(1 + z_0^2)(M^2 + z_0^2)}{2z_0\; 2(M^2 + z_0^2) - M(1 + z_0^2)}.$$

These estimates are sharp.

P r o o f. We are looking for extremal values of linear functionals over a convex and compact set, so it is sufficient to consider corresponding problems for extreme points. Since these points are of the form (3), the theorem follows immediately.

REFERENCES

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STRESZCZENIE

W pracy rozważa się pewną klasę funkcji typowo-rzeczywistych i ograniczonych w kole jednostkowym, mających unormowanie Montela $f(0) = 0$, $f(z_0) = z_0$, $z_0 \in (-1,1)$, danych wzorem strukturalnym

$$f(z) = \int_{-1}^{1} G(z,t)\,d\mu(t) .$$

W klasie tej wyznaczono obszar zmienności $f(z)$ oraz dokładne oszacowania od góry i od dołu dla $|a_1|$, $|a_2|$.

Резюме

В работе рассматривается некоторый подкласс типично-вещественных и органиченных функций в единичном круге имеющих нормирование Монтеля $f(0)=0$, $f(z_0)=z_0$, $z_0\in(-1,1)$ и даных структурной формулой $f(z)=\int_{-1}^{1} G(z,t)\,d\mu(t)$

В этом классе определено область изменения $f(z)$, а также точные оценки с верху и снизу для $/a_1/$, $/a_2/$.