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Note on Iterations of Some Entire Functions

Uwaga o iteracjach pewnych funkcji całkowitych

Заметка об итерациях некоторых целых функций

Let f be entire or rational function. Consider the sequence of iterations

$$f_0(z) = z, \quad f_{n+1}(z) = f(f_n(z)), \quad n = 0,1,\dots .$$

In the iteration theory an important part is played by the set $F(f)$ of those points of the complex plane $\mathbb{C}$ where $\{f_n\}$ is not normal in the sense of Montel. It is well known that the set $F(f)$ has the following properties (cf. [2], [4], [5], [7])

1) $F(f)$ is nonempty and perfect.

2) $F(f_n) = F(f)$ for $n>1$.

3) $F(f)$ is completely invariant with respect to f, i.e., for every β, $\beta \in F(f) \Leftrightarrow f(\beta) \in F(f) \cap f^{-1}(\{\beta\}) \subset F(f)$.

A point α is said to be a fixed point of order n iff $f_n(\alpha) = \alpha$ and $f_k(\alpha) \neq \alpha$ for $k = 1,2,\dots,n-1$.

The derivative $f_n'(\alpha)$ is called a multiplier of fixed point α. A fixed point of order n is called attractive, indifferent or repulsive according as

$$|f_n'(\alpha)| < 1, \quad |f_n'(\alpha)| = 1, \quad |f_n'(\alpha)| > 1,$$

respectively.

4) Every repulsive fixed point belongs to $F(f)$ and every attractive fixed point does not belong to $F(f)$.

It is also known that if f is rational and $F(f)$ has a nonempty interior then $F(f) = \mathbb{C}$. In 1918 Latte constructed a rational function for which this case really occurs (cf. also [3]).

I.N. Baker [1] proved that there is a $k > e^2$ such that $F(kze^z) = \mathbb{C}$. However, the question if $F(e^z) = \mathbb{C}$ is still open.

The aim of this paper is to prove the following

THEOREM. $F(2k\pi ie^z) = \mathbb{C}$, $k = \pm 1, \pm 2, \ldots$.

Let f be entire, let S denote the set of all finite singular points of the function f^{-1} and put

$$E(f) = \bigcup_{n=0}^{\infty} f_n(S).$$

In the sequel D is a domain contained in $\mathbb{C} \setminus F(f)$.

We shall use the following results proved by I.N. Baker [1].

THEOREM 1. If $\lim_{k\to\infty} f_{n_k}(z) = \alpha$, $z \in D$, $\alpha \in \mathbb{C}$, then $\alpha \in L(f) := \overline{E(f)} \cup \{\infty\}$.

THEOREM 2. If int $L = \emptyset$ and $\mathbb{C} \setminus L$ is connected then for every convergent subsequence $\{f_{n_k}\}$ of iterates

$$\lim_{k \to \infty} f_{n_k}(z) = \alpha(z), \quad z \in D \Rightarrow \alpha(z) = \text{const.}$$
$$\text{for } z \in D.$$

P r o o f of the Theorem. Put $f(z) = 2k\pi i e^z$ and note that for the inverse function f^{-1} the point $z = 0$ is the unique singularity which is transcendental. Hence the set $L = L(f)$ has the form

$$L = \{0, 2k\pi i, \infty\}.$$

Since int $L = \emptyset$ and $\mathbb{C} \setminus L$ is connected, by Theorems 1 and 2, every limit function of any convergent subsequence of $\{f_n\}$ in D is constant and equals to 0, $2k\pi i$ or ∞.

Now we shall show that:

∞ is not a limit of any subsequence $\{f_{n_k}\}$ in D.

For an indirect proof suppose that there is a subsequence $\{f_{n_k}\}$ and a domain D such that $\lim_{k \to \infty} f_{n_k}(z) = \infty$ for $z \in D$. Let us note that this implies

$$\lim_{n \to \infty} f_n(z) = \infty \quad \text{for } z \in D.$$

Indeed, in the opposite case one can find another subsequence $\{f_{\bar{m}_k}\}$ which converges to one of the remaining points of the set L for $z \in D$. Hence for every compact set $K \subset D$ there are an $a > 2k\pi$ and infinitely many n such that

$$f_n(K) \subset \{z : |z| \leqslant a\}.$$

Because $|f(a)| < |f(|f(a)|)|$, we have

$$f_n(K) \subset \{z : |z| > |f(|f(a)|)|\} \not\supset f_{n-1}(K).$$

for infinitely many n. Evidently, for such an n,

$$f_{n-1}(K) \not\subset \{z : |z| < |f(a)|\}.$$

Therefore, for infinitely many n we have

$$f_{n-1}(K) \cap B \neq \emptyset,$$

where $B := \{z : |f(a)| \leq |z| \leq |f(|f(a)|)|\}$.
Consequently, one can find a subsequence of $\{f_n\}$ which converges to a point of the set B. Since $B \cap L = \emptyset$ this is a contradiction. Thus we have proved that

$$\lim_{n\to\infty} f_n(z) = \infty, \qquad z \in D.$$

The function $f(z) = 2k\pi ie^z$ is bounded in the left half plane $\omega = \{z : \operatorname{Re} z \leq 0\}$. Therefore

$$f_n(K) \cap \omega = \emptyset$$

for sufficiently large n. In particular, for those n,

$$f_n(K) \cap R_- = f_n(K) \cap f^{-1}(R_-) = \emptyset$$

where $R_- := (-\infty, 0)$. One can easily verify, that $f^{-1}(R_-)$ consists of the straight lines $y = \frac{\pi}{2} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ The complement of the set $f^{-1}(R_-)$ does not contain a disc of diameter greater than 2π. On the other hand we have

$$f'[f_n(z)] = f_{n+1}(z)$$

and consequently

$$\lim_{n\to\infty} f'[f_n(z)] = \infty$$

uniformly in the compact sets $K \subset D$. Take a compact set $K \subset D$

with int $K \neq \emptyset$ and $z_0 \in \text{int } K$. Hence

$$\lim_{n \to \infty} f_n'(z_0) = \lim_{n \to \infty} \prod_{j=0}^{n-1} f'[f_j(z_0)] = \infty$$

and there is an $r > 0$ such that $U_{z_0} = \{z : |z - z_0| < r\} \subset K$. The functions

$$g_n(z) := \frac{f_n(z)}{f_n'(z_0)}, \qquad n = 1,2,\ldots,$$

are holomorphic in the disc U_{z_0}. By Bloch's theorem ([6], p. 386) there exists a disc $U_n(b)$ of positive radius b such that

$$U_n(b) \subset g_n(U_{z_0}) = \frac{f_n(U)}{f_n'(z_0)}, \qquad n = 1,2,\ldots .$$

i.e., $f_n'(z_0)U_n(b) \subset f_n(U_{z_0}) \subset f_n(K)$, $n = 1,2,\ldots$. The diameter of the set $f_n'(z_0)U_n(b)$ is equal to $2|f_n'(z_0)|b$ and is greater than 2π for $n \geq n_0$. This implies that

$$f_n(K) \cap f^{-1}(R) \neq \emptyset \qquad \text{for } n \geq n_0$$

which is impossible. This contradiction proves that ∞ cannot be a limit of any subsequence of $\{f_n\}$.

In the sequel we shall need the following

LEMMA. If $L = L(f)$ is closed and consists of isolated points then every repulsive fixed point α of the function f is not a limit of any subsequence of $\{f_n\}$.

P r o o f of the Lemma. By assumption $A := |f'(\alpha)| > 1$. Take an $\varepsilon > 0$ such that $A - \varepsilon > 1$. There is a $\delta > 0$ such that

$$(A-\varepsilon)|z-\alpha| < |f(z)-\alpha| < (A+\varepsilon)|z-\alpha|$$

for

$$|z-\alpha| < (A+\varepsilon)^2\delta$$

and

$$(A+\varepsilon)^2\delta < \inf\{|\alpha-\beta| : \beta\in L, \quad \beta\neq\alpha\}.$$

Suppose that

$$\lim_{k\to\infty} f_{n_k}(z) = \alpha, \qquad z\in D.$$

Hence, for compact $K\subset D$ we have

$$f_n(K)\subset\{z : |z-\alpha| < (A-\varepsilon)\delta\}$$

for infinitely many n. Since

$$|f(z)-\alpha| > (A-\varepsilon)|z-\alpha| > |z-\alpha|$$

for $|z-\alpha| < (A+\varepsilon)^2\delta$, we have

$$f_{n+1}(K)\not\subset\{z : |z-\alpha| < (A-\varepsilon)\delta\}$$

and

$$|f_{n+1}(z)-\alpha| < (A+\varepsilon)|f_n(z)-\alpha| < (A+\varepsilon)(A-\varepsilon)\delta <$$
$$< (A+\varepsilon)^2\delta$$

for the same n. Putting

$$B = \{z : (A-\varepsilon)\delta \leq |z-\alpha| \leq (A+\varepsilon)^2\delta\}$$

we see that $B\cap f_n(K)\neq\emptyset$ for infinitely many n. Consequently, there exists a subsequence of $\{f_n\}$ which has a limit in B. By Theorem 1 and 2 this is a contradiction, because

$B \cap \{L \setminus \{\alpha\}\} = \emptyset$. This completes the proof of the Lemma.

It is easily seen that $z = 2k\pi i$ is a repulsive fixed point of f. By Lemma, $2k\pi i$ cannot be a limit of any subsequence of $\{f_n\}$.

Supposing that $\lim_{k \to \infty} f_{n_k}(z) = 0$ for $z \in D$, we see that

$$\lim_{k \to \infty} f_{n_k+1}(z) = f(0) = 2k\pi i.$$

This contradicts the previous part of proof and completes the proof of the Theorem.

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STRESZCZENIE

W teorii iteracji funkcji całkowitych f podstawową rolę odgrywa zbiór F(f) tych punktów płaszczyzny w których ciąg iteracji f_n funkcji f nie jest rodziną normalną w sensie Montela.

W tej pracy dowodzi się, że $F(2k\pi ie^z)$, $k = \pm 1, \pm 2, \ldots$ jest całą płaszczyzną.

Резюме

В теории итерации целых функций f основную роль играет множество $F(f)$ этих точек, в которых последовательность итерации функции f_n функции f не является нормальным семейством в смысле Монтеля. В этой работе доказывается, что $F(2k\pi ie^z)$ $k = \pm 1, \pm 2, \ldots$ составляет целую плоскость.