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Coefficients of Inverses of Regular Starlike Functions

Współczynniki funkcji odwrotnych do funkcji regularnych gwiaździstych

Коэффициенты функций обратных к регулярным звездным функциям

1. INTRODUCTION

As is usually the case we let $\mathcal{S}$ represent the class of functions of the form

$$(1.1) \qquad f(z) = z + a_2z^2 + a_3z^3 + \dots$$

regular and univalent in the open unit disk $\Delta = \{z \in \mathbb{C} : |z|<1\}$ Much of the interest in and many investigations of $\mathcal{S}$ relate to establishing correct bounds on the coefficients a_k, $k = 2,3,\dots$, and it has been shown, cf.e.g. [2], that $|a_n| \leq n$, for $n = 2,3,4,5,6$. Except for rotations the unique extremal for these bounds is the Koebe function

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$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots \tag{1.2}$$

In his seminal work relating to the conclusion that $|a_3| \leq 3$, Loewner [7] was able to give sharp bounds for the coefficients which appear in the Maclaurin series for the inverse of any function in $\mathcal{S}$. Specifically, if the inverse of $f(z)$ is

$$F(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \dots \tag{1.3}$$

he showed that

$$|\gamma_n| \leq \frac{1}{n}\binom{2n}{n+1} \tag{1.4}$$

for $n \geq 2$ and that the sharp upper bound is achieved by the inverse of a rotation of $k(z)$ defined by (1.2).

To summarize the situation briefly we can say that sharp bounds for $|\gamma_n|$ and each index n have been obtained in a surprisingly straightforward way, whereas proper bounds on $|a_n|$ have been obtained for only a few indices with great difficulty. The purpose of this note is to illustrate that the converse situation appears to hold for some well-known subclasses of S.

2. CONCLUSIONS

For $0 \leq \alpha \leq 1$ we let $\mathcal{S}^*_\alpha$ be the subclass of $\mathcal{S}$ consisting of functions which are α-starlike, i.e., $f(z)$ is as in (1.1) and $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$ for z in Δ. The functions $f(z)$ in $\mathcal{S}$ for which $f[\Delta]$, the image of Δ under $f(z)$, is a convex domain is denoted by K; it is well-known that $K \subset S^*_{\frac{1}{2}}$.

The family of all starlike functions is $\mathcal{S}^*_0$, written simply as $\mathcal{S}^*$.

Also, let $\mathcal{P}$ be the class of functions

$$(2.1) \qquad P(z) = 1 + p_1 z + p_2 z^2 + \dots$$

regular and satisfying the condition $\operatorname{Re} P(z) > 0$ for z in Δ. It follows that $f(z)$ is in $\mathcal{S}^*_\alpha$ if and only if there is a corresponding function $P(z)$ in $\mathcal{P}$ for which

$$(2.2) \qquad zf'(z) = f(z)\,(1 - \alpha)P(z) + \alpha$$

With representations (1.1) and (2.1) the last relation yields the relationships

$$(2.3) \qquad (n - 1)a_n = (1 - \alpha)\sum_{j=1}^{n-1} p_j a_{n-j-1}, \qquad n = 2,3,\dots .$$

Now, if a function and its inverse are given by (1.2) and (1.3) a brief calculation shows that

$$(2.4) \qquad \gamma_2 = -a_2, \quad \gamma_3 = 2a_2^2 - a_3 \quad \text{and} \quad \gamma_4 = 5a_2[a_3 - a_2^2] - a_4$$

and these along with (2.3) give $\gamma_2 = -(1 - \alpha)p_1$ and

$$(2.5) \qquad \gamma_3 = -\left(\frac{1-\alpha}{2}\right)\left[p_2 - 3(1 - \alpha)p_1^2\right],$$

which give rise to the following result.

THEOREM 1. <u>If</u> $f(z)$ <u>is in</u> $\mathcal{S}^*_\alpha$ <u>and its inverse is given by</u> (1.3), <u>then</u> $|\gamma_2| \leq 2(1 - \alpha)$ <u>and</u>

$$(2.6) \qquad |\gamma_3| \leq \begin{cases} (1 - \alpha)(5 - 6\alpha) & \text{for } 0 \leq \alpha \leq \frac{2}{3}, \\ (1 - \alpha) & \text{for } \frac{2}{3} \leq \alpha < 1. \end{cases}$$

These bounds are sharp.

The first bound follows from the relation $|p_k| \leq 2$ which is valid for all coefficients of (2.1) and the second is a consequence of the following lemma which is quoted in [6].

LEMMA. If $P(z)$ in $\mathcal{P}$ is given by (2.1), then

$$|p_2 - \mu p_1^2| \leq 2 \max\{1, |1 - 2\mu|\} \tag{2.7}$$

and the bound is rendered sharp by $Q(z) = (1 + z)/(1 - z)$ when $|1 - 2\mu| \geq 1$ and by $T(z) = (1 + z^2)/(1 - z^2)$ otherwise.

Now, replacing $P(z)$ in (2.2) by $Q(z)$ and $T(z)$ and solving for the corresponding $f(z)$ gives functions in $\mathcal{S}_\alpha^*$, namely

$$k_\alpha(z) = \frac{z}{(1 - z)^{2(1-\alpha)}} = z + 2(1 - \alpha)z^2 + (1 - \alpha)(3 - 2\alpha)z^3 + \ldots \tag{2.8}$$

and

$$h_\alpha(z) = \frac{z}{(1 - z^2)^{1-\alpha}} = z + (1 - \alpha)z^3 - \frac{\alpha(1 - \alpha)}{2} z^5 + \ldots , \tag{2.9}$$

respectively. Appealing to (2.4) we see that $k_\alpha(z)$ gives the sharp upper bound for $|\gamma_2|$ with any value of α and for $|\gamma_3|$ when $0 \leq \alpha \leq \frac{2}{3}$, whereas $h_\alpha(z)$ provides equality in (2.6) for the remaining values of α.

Theorem 1 shows that no single function serves as the extremal for all coefficients γ_n of inverses for members of $\mathcal{S}_\alpha^*$, $\frac{2}{3} \leq \alpha < 1$, which differs significanty from $\mathcal{S}$ where one function can provide all extremal values. The situation for K appears to be surprisingly difficult; (2.8) with $\alpha = \frac{1}{2}$ gives sharp upper bounds $|\gamma_2| \leq 1$ and $|\gamma_3| \leq 1$ when f(z) is in K however $k_{\frac{1}{2}}(z)$ cannot give the sharp upper bound for $|\gamma_n|$ for all n. Furthermore it is not likely that using (2.3) and (2.4) and the methods of the theorem can provide the correct bound for γ_4. However, we can provide an estimate for $|\gamma_n|$.

THEOREM 2. If $F(w) = w + \gamma_2 w^2 + \ldots$ corresponds to f(z) in $\mathcal{S}^*_\alpha$, then

$$|\gamma_n| \leq \frac{1}{n} \frac{\Gamma(2n(1-\alpha)+1)}{[\Gamma(n(1-\alpha)+1)]^2}. \tag{2.10}$$

To establish (2.10) we represent γ_n in a novel way, cf. [5]. Let f(z) and F(w) be as in (1.1) and (1.3) and let c(r) be the image of $\{|z| = re^{i\theta} ; 0 \leq \theta \leq 2\}$ under f(z), then

$$\gamma_n = \frac{1}{2\pi i} \int_{c(r)} \frac{F(w)dw}{w^{n+1}} = \frac{1}{2\pi i} \int_{|z|=r} \frac{zf'(z)}{f(z)^{n+1}} dz =$$

$$= \left(\frac{1}{2\pi i}\right)\left(\frac{-1}{n}\right)\left\{\frac{z}{f(z)^n}\Bigg|_{|z|=r} - \int_{|z|=r} \frac{dz}{f(z)^n}\right\} = \tag{2.11}$$

$$= \frac{1}{2\pi i n} \int_{|z|=r} \frac{dz}{f(z)^n}.$$

Now, if f(z) belongs to $\mathcal{S}^*_\alpha$, it is known [4] that $\left(\frac{z}{f(z)}\right)^{\frac{1}{2(1-\alpha)}} = 1 + \omega(z)$, where $|\omega(z)| \leq |z|$.
Consequently, using (2.11) and the principle of subordination we may write

$$|\gamma_n| = \frac{1}{2\pi n}\left|\int_{|z|=r}\left(\frac{z}{f(z)}\right)^n \frac{dz}{z^n}\right|$$

(2.12)
$$\leq \frac{1}{2\pi n r^n}\int_{|z|=r}|1+\omega(z)|^{2n(1-\ \ )}|dz|$$

$$\leq \frac{1}{2\pi n r^n}\int_{|z|=r}|1+z|^{2n(1-\alpha)}|dz|.$$

Letting $z = re^{i\theta}$ and replacing r by 1 gives

$$|\gamma_n| \leq \frac{2^{2n(1-\alpha)}}{2\pi n}\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|^{2n(1-\alpha)}d\theta =$$

(2.13)
$$= \frac{2^{2n(1-\alpha)}}{2\pi n}\int_0^{\frac{\pi}{2}}(\cos t)^{2n(1-\alpha)}dt =$$

$$= \frac{1}{n}\,\frac{\Gamma(2n(1-\alpha)+1)}{[\Gamma(n(1-\alpha)+1)]^2},$$

having made reference to standard tables, [3] for example.

For $\alpha = 0$, (2.13) gives $|\gamma_n| \leq \frac{1}{n}\binom{2n}{n} = B_n$ which exceeds the correct value given in (1.4). However the orders of both bounds, as $n \to \infty$, are the same. Also, for $\alpha = 0$, the computations given in (2.12) and (2.13) are equivalent to computing an upper bound for $|\gamma_n|$ when $f(z)$ is the Koebe function (1.2); hence it follows from the work of Baernstein [1] that B_n is an upper bound for coefficients of functions in $\mathcal{S}$. Of course, this is superfluous in view of Loewner's earlier result, namely (1.4), but it does provide the correct order for $|\gamma_n|$, $n \to \infty$, with relative ease.

It appears then, that bounds for $|\gamma_n|$, $f(z)$ in $\mathcal{S}^*_\alpha$ $\alpha \neq 0$, or $f(z)$ in K may be obtainable only with considerable difficulty and that no single member of the class provides a sharp bound for all indices; on the other hand good bounds for $|a_n|$ are obtainable in a straight forward fashion [2].

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STRESZCZENIE

Otrzymano ostre oszacowania początkowych współczynników dla funkcji odwrotnych do funkcji α-gwiaździstych oraz oszacowania nieostre dla wszystkich współczynników.

Резюме

В работе получено строгие оценки начальных коэффициентов для функций обратных к α-звездным функциям, а также оценки слабые для всех коэффициентов.