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On Univalence of a Certain Integral

O jednolistności pewnej całki

Об однолистности некоторого интеграла

INTRODUCTION

Suppose that α, β are fixed complex numbers, $f(z) = z + \dots$ and $g(z) = z + \dots$ are functions analytic in the unit disk Δ and $f'(z)\frac{g(z)}{z} \neq 0$ for z in Δ. We shall be concerned here with a function $h(z)$ defined by the formula:

$$h(z) = \int_0^z \left[f'(t)\right]^\alpha \left[\frac{g(t)}{t}\right]^\beta dt \tag{1.1}$$

Our aim is to establish some sufficient conditions for univalence of $h(z)$. To do this we need some results which we quote below.

L. Ahlfors showed [1] that if c is a complex number and $|c| \leq k < 1$ for a given k, then any function $b(z)$ analytic in Δ which satisfies the condition

$$(1.2)\qquad \left|(1-|z|^2)\frac{zb''(z)}{b'(z)} + c|z|^2\right| \leq k = \frac{K-1}{K+1} \qquad z \in \Delta$$

is univalent in Δ and it has a K-quasiconformal extension to the whole plane.

Following Ch. Pommerenke [5] we denote by ord(f),

$$(1.3)\qquad \operatorname{ord}(f) = \sup_{\zeta\in\Delta}\left|-\bar{\zeta} + \frac{1-|\zeta|^2}{2}\frac{f''(\zeta)}{f'(\zeta)}\right|,$$

the order of the linearly invariant family generated by a locally univalent function f(z).

It is known that $\operatorname{ord}(f) \geq 1$ with equality to hold if and only if f(z) is univalent and it maps Δ onto a convex domain [5]. Moreover, $\operatorname{ord}(f) \leq 2$ for a univalent f.

MAIN RESULTS

We start with the following

THEOREM 1. Suppose that: (i) f(z) is analytic and locally univalent in Δ such that $\operatorname{ord}(f) = A < \infty$, (ii) g(z) is analytic and univalent in Δ, (iii) α, β are complex numbers subject to the conditions

$$|\alpha| < \tfrac{1}{2}, \qquad 2|\alpha|A + 4|\beta| \leq 1.$$

Then h(z) defined by (1.1) is univalent in Δ.
Moreover, if $2|\alpha|A + 4|\beta| < 1$, then h(z) has a quasiconformal extention to the whole plane.

P r o o f. We are going to make use of the condition (1.2). It will follow from the method of proof that best choice for c is $c = -2\alpha$.

We have

$$(*)\quad \left|(1-|z|^2)\frac{zh''(z)}{h'(z)} + c|z|^2\right| \leq 2|\alpha||z|\left|\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \bar{z}\right| + |\beta|(1-|z|^2)\left|\frac{zg'(z)}{g(z)} - 1\right|$$

for $z \in \Delta$.

Since $|c| < 1$ we have $|\alpha| < \frac{1}{2}$. Univalence of $g(z)$ implies $(1-|z|^2)\left|\frac{zg'(z)}{g(z)} - 1\right| < 4$, $(|z| < 1)$ which we combine with $(*)$ to obtain

$$\left|(1-|z|^2)\frac{zh''(z)}{h'(z)} - 2\alpha|z|^2\right| < 2|\alpha|A + 4|\beta|$$

By the result of Ahlfors the function $h(z)$ is univalent in Δ if the following conditions are fulfilled

$$2|\alpha|A + 4|\beta| < 1 \quad \text{and} \quad |\alpha| < \frac{1}{2}.$$

If $2|\alpha|A + 4|\beta| = 1$, then by applying (1.2) to the function $h_r(z) = h(rz)$, $z \in \Delta$, $r \in (0,1)$ one gets

$$\left|(1-|z|^2)\frac{zh_r''(z)}{h_r'(z)} - 2\alpha|z|^2\right| \leq 2|\alpha|A(r|z| - 1) + 1 < 1$$

for z in Δ.

This shows univalence of $h_r(z)$ in Δ which implies univalence of $h(z)$ in Δ.

Note that if $2|\alpha|A + 4|\beta| = 1$, then $h(z)$ may not have a quasiconformal extension to the whole plane.

COROLLARY 1. If $f(z)$ and $g(z)$ are both analytic and univalent in Δ, if $|\alpha| + |\beta| \leq \frac{1}{4}$, then $h(z)$ is univalent in Δ.

P r o o f. This follows from Th. 1. in view of the well-known inequality $\mathrm{ord}(f) \leq 2$ for univalent functions.

COROLLARY 2. If $\beta = 0$ our result reduces to a theorem of J. Pfaltzgraff [4].

If $\alpha = 0$ we obtain a generalization of a result due to W.M. Causey [2]. He showed that $\int_0^z \left[\frac{g(z)}{t}\right]^\beta dt$ is univalent for complex β, $|\beta| \leq \frac{\sqrt{2}-1}{4} \approx 0,102 \ldots$. Our Corollary extends this results to complex β : $|\beta| \leq \frac{1}{4} = 0,25$.

THEOREM 2. If $f(z)$, $g(z)$ are functions close - to - convex in Δ and $0 < \alpha + \beta \leq 1$, $\alpha, \beta \geq 0$ - real, then $h(z)$ is also close - to - convex in Δ.

P r o o f. It is well - known, that if $g(z)$, $f(z)$ are close - to - convex, then

$$f'(z) = \varphi'(z)\, p(z) \tag{2.1}$$

$$\frac{g(z)}{z} = G'(z) \tag{2.2}$$

where $\varphi(z)$ is univalent and convex in Δ, $G(z)$ is close- - to - convex while $p(z)$ is analytic and such that $\mathrm{Re}\, p(z) > 0$. Moreover, for some convex fonction Φ and $P(z)$ of positive real part

$$G'(z) = \Phi'(z)\, P(z).$$

Thus we have

$$h'(z) = f'^{\alpha}(z)\left(\frac{g(z)}{z}\right)^{\beta} = (\varphi'^{\alpha}(z)\Phi'^{\beta}(z))(p^{\alpha}(z)P^{\beta}(z)) =$$

$$= w'(z)\, D(z)$$

where

$$w'(z) = \varphi'^{\alpha}(z)\Phi'^{\beta}(z),\ D(z) = p^{\alpha}(z)\, P^{\beta}(z).$$

It is easy to see that $w(z)$ is a function univalent and convex provided α, β are real and such that $\alpha + \beta \leq 1$, and that $\operatorname{Re} D(z) \geq 0$ provided $|\alpha| + |\beta| \leq 1$. Ultimately $h(z)$ is a close - to - convex function if α, β are real and subject to the condition $0 < \alpha + \beta \leq 1$.

The previous considerations gave us a set of values of α, β for which the integral (1.1) is a univalent function in Δ.

We now want to find a set of values of α, β for which that integral is not univalent. To do this we give some examples. First we recall a result due to W. Royster which we state as.

LEMMA [6]. <u>The function</u> $F(z) = \exp[\nu \log(1 - z)]$ <u>is univalent in</u> Δ <u>if and only if</u> ν <u>satisfies one of the conditions</u>

(2.4) $$|\nu + 1| \leq 1, \qquad |\nu - 1| \leq 1.$$

We now prove

THEOREM 3. <u>Let the functions</u> $f(z)$, $g(z)$ <u>defined by the formulas</u>

$$f(z) = \exp[\mu \log(1 - z)]$$

$$g(z) = z \cdot \exp\left[(i - 1)\log(1 - z)\right]$$

be univalent in Δ.

If complex numbers α, β satisfy the conditions

(i) $|\alpha| > 1 - |\beta|\sqrt{2}$ and $\beta \neq 1 + i$

or

(ii) $|\alpha| > 1 - |2\alpha + \beta(1 - i)|$ and $\alpha \neq 1 - \beta(\frac{1}{2} - \frac{i}{2})$

or

(iii) $|\beta| \geq \frac{1}{\sqrt{2}}$ or $|2\alpha + \beta(1 - i)| \geq 1$,

then $h(z)$ given by

$$h(z) = \int_0^z f'^{\alpha}(t)\left(\frac{g(t)}{t}\right)^{\beta} dt$$

does not belong to S.

Proof. If $f(z)$, $g(z)$ are as stated, then

$$h(z) = A_1 \exp\left\{\left[\alpha(\mu - 1) + \beta(i - 1) + 1\right]\log(1 - z)\right\} + A_2$$

In view of the Lemma $h(z)$ is univalent if and only if the point $\nu = \alpha(\mu - 1) + \beta(i - 1) + 1$ lies in one of the disks (2.4).

Hence, $h(z)$ is not univalent if and only if μ satisfies (2.4) and ν belongs to the set N.

$$N = \{\nu : |\nu + 1| > 1\} \cap \{\nu : |\nu - 1| > 1\}$$

Since $\mu = 1 + \frac{\nu - 1 + \beta(1 - i)}{\alpha}$ the conditions (2.4) are equivalent to

$$|\nu - 1 + \beta(1 - i)| \leq |\alpha|$$

$$|\nu - 1 + 2\alpha + \beta(1 - i)| \leq |\alpha|$$

Our statement will be proved if we find such relationships between α , β that the intersection of N and at least one of the disks (2.4) is non - empty.
This leads to (i), (ii), (iii).

COROLLARY 3. [6] . If $\beta = 0$, then $\int_0^z (f'(t))^\alpha dt$ is not univalent for any complex α such that $|\alpha| > \frac{1}{3}$ and $\alpha \neq 1$.

COROLLARY 4. If $\alpha = 0$, then $\int_0^z \left[\frac{g(t)}{t}\right]^\beta dt$ is not univalent for any complex β such that $|\beta| > \frac{1}{\sqrt{2}} \approx 0,709$.

REMARK. W.M. Causey [3] showed that $\int_0^z \left[\frac{g(t)}{t}\right]^\beta dt$ is not univalent for real β , $\beta > 0,5$.
Corollary 4 partly extends this result for complex β , $|\beta| > \frac{1}{\sqrt{2}}$.

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STRESZCZENIE

Niech f, g oznaczają funkcje analityczne w kole jednostkowym D i takie, że $f'(z)g(z)z^{-1} \neq 0$. Celem pracy jest ustalenie warunków koniecznych lub dostatecznych na stałe α, β oraz funkcje f, g przy których funkcja h określona wzorem (1.1) jest jednolistna w D.

Резюме

Пусть f, g обозначают аналитические функции в круге D, и такие что $f'(z)g(z)z^{-1} \neq 0$. Цель этой работы - это определение необходимых или достаточных условий на α, β и на f, g, при которых функция дана формулой (1,1) однолистная в D.