ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXIV, 6 SECTIO A 1980

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On Distributions and Moments of Order Statistics for Random Sample Size

O rozkładach i momentach statystyk porządkowych dla próby o losowej liczebności

О распределениях и моментах порядковых статистик для выборки случайного объема

1. Introduction. Let $(X_1, X_2, .., X_n)$ be a sample from a population having distribution function (d.f.) $\bar{F}(x)$ and probability density function (p.d.f.) $\bar{f}(x)$. Suppose that $X_1 \leqslant \leqslant X_2 \leqslant ... \leqslant X_n$ is an ordered sample of size n. Properties of order statistics (o.s.) for fixed sample size n were widely investigated, while a literature on this subject in the case when n is a value of random variable N is not so rich. Some studies on order statistics for random sample size can be found, e.g. in [1], [2] and [6].

In this note, we will study distributions and moments of order statistics in the case when N has a power series distribution (e.g. [3], [5]) and an inflated power series distribution. In particular cases, some of our result reduces to those of [6]. It is worth mentioning here that such types of distributions are of interest in mathematical statistics ([4], [7], [8], [9], [10]).

2. N has a power series distribution.

(i) Distributions of order statistics.

A random variable N is said to have the power series distribution (PSD), if the probability function of N is of the form

$$p(k;\theta) = P[N=k] = \frac{a(k)\,\theta^k}{\bar{f}(\theta)} \quad \text{for } k \in T, \tag{1}$$

where $T \in \mathbf{N} \cup \{0\}$, $a(k) > 0$, $\bar{f}(\theta) = \sum_{k \in T} a(k)\,\theta^k$ for $\theta \in \Omega = \{\theta : 0 < \theta < \rho\}$, the parameter space, and ρ is the radius of convergence of the power series of $\bar{f}(\theta)$, and $\mathbf{N}$ denotes the set of all integers.

In what follows we write f_i for $f(x_i)$, F_j for $F(x_j)$ etc., and put

$$A_i(\theta) = \sum_{\substack{k > i\\ k \in T}} a(k)\,\theta^k,\quad D_i(\theta, F) = \sum_{\substack{k \in T\\ k > i}} \binom{k}{i} a(k)\,[\theta(1-F_i)]^k,$$

$$E_i(\theta, F) = \sum_{\substack{k > i\\ k \in T}} \binom{k}{i} a(k)\,[\theta F_{N-i+1}]^k,$$

$$L_j(\theta, F) = \sum_{k > j} k(k-1)\binom{k-2}{i-1,\ j-i-1,\ k-j} a(k)\,[\theta(1-F_j)]^k,$$

where $\binom{n}{k_1, k_2, k_3}$ is the multinomial coefficient.

Lemma 1. *If $X_1 \leq X_2 \leq \dots \leq X_N$ is an ordered sample of size N, where N has the power series distribution* (1), *then:*

a) *the conditional* p.d.f. *of X_i, $1 \leq i \leq N$, conditioned on the event $[N \geq i]$ is*

$$g(x_i) = \frac{D_i(\theta, F)}{A_i(\theta)}\, i F_i^{i-1} (1-F_i)^{-i} f_i, \tag{2}$$

b) *if T is finite, then the conditional* p.d.f. *of X_{N-i+1}, conditioned on the event $[N \geq i]$ is*

$$g(x_{N-i+1}) = \frac{E_i(\theta, F)}{A_i(\theta)}\, i(1-F_{N-i+1})^{i-1} F_{N-i+1}^{-i} f_{N-i+1}, \tag{3}$$

c) *the joint conditional* p.d.f. *of X_i and X_j, $1 \leq i < j \leq N$, conditioned on the event $[N \geq j]$ is*

$$g(x_i, x_j) = \frac{L_j(\theta, F)}{A_j(\theta)} F_i^{i-1} [F_j - F_i]^{j-i-1} [1-F_j]^{-j} f_i f_j. \tag{4}$$

Proof. Let G denote the conditional cumulative distribution function of X_i, conditioned on the event $[N \geq i]$, i.e. $G(x_i) = P[X_i < x_i \mid N \geq i]$. For a given sample size k let us put $H(x_i \mid k) = P[X_i < x_i \mid N = k]$ and $h(x_i \mid k) = H'(x_i \mid k)$. We have

$$G(x_i) = P[X_i < x_i \mid N \geq i] = \frac{1}{P[N \geq i]} \sum_{\substack{k > i\\ k \in T}} P[X_i < x_i \mid N = k] =$$

$$= \frac{1}{P[N \geq i]} \sum_{\substack{k > i\\ k \in T}} H(x_i \mid k)\, P[N = k].$$

Hence, we get

$$g(x_i) = \frac{1}{P[N \geq i]} \sum_{\substack{k \geq i \\ k \in T}} h(x_i \mid k) P[N = k] . \tag{5}$$

By (1), we have

$$\sum_{\substack{k \geq i \\ k \in T}} h(x_i \mid k) P[N = k] = \sum_{\substack{k \geq i \\ k \in T}} \frac{k!}{(i-1)!\,(k-i)!} F_i^{i-1} (1 - F_i)^{k-i} \frac{a(k)\,\theta^k}{\bar{f}(\theta)} f_i =$$

$$= \frac{i F_i^{i-1} (1 - F_i)^{-i} f_i}{\bar{f}(\theta)} D_i(\theta, F) ,$$

since $h(x_i \mid k) = \dfrac{k!}{(i-1)!\,(k-i)!} F_i^{i-1} (1 - F_i)^{k-i} f_i$, $i = 1, 2, ..., k$. Moreover,

$$P[N \geq i] = \frac{A_i(\theta)}{\bar{f}(\theta)} .$$

Hence we obtain (2).

The formulae (3) and (4) can be obtained in an analogous way, using

$$h(X_{N-i+1} \mid k) = \frac{k!}{(i-1)!\,(k-i)!} (1 - F_{N-i+1})^{i-1} F_{N-i+1}^{k-i} f_{N-i+1}$$

and

$$h(x_i, x_j \mid k) = \frac{k!}{(i-1)!\,(j-i-1)!\,(k-j)!} F_i^{i-1} [F_j - F_i]^{j-i-1} [1 - F_j]^{k-j} f_i f_j$$

for $i < j$.

The p.d.f. of the smallest and largest (when T is finite) order statistics are directly obtained from (2) and (3) by putting $i = 1$.

Lemma 2. *Let R be the range of an ordered sample $X_1 \leq X_2 \leq ... \leq X_N$, where N is a random variable distributed according to* (1). *Then*

$$g(R) = \frac{2}{A_2(\theta)} \sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k)\,\theta^k \int_{-\infty}^{\infty} [F(x + R) - F(x)]^{k-2} f(x) f(x + R)\, dx . \tag{6}$$

Proof. Since the p.d.f. of the range R ([11] p. 248) is, for fixed k,

$$h(R) = k(k-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{k-2} f(x) f(x+R)\, dx,$$

therefore the p.d.f. of R for random N with the distribution (1) is given by (6).

(ii) A sample from a population uniformly distributed in (0; 1).

We now consider a sample $(X_1, ..., X_N)$ of size N from a population having the uniform distribution in (0; 1), i.e. $F(x) = x$ for $x \in (0, 1)$. In this case we can prove

Theorem 1. *If $X_1 \leqslant X_2 \leqslant ... \leqslant X_N$ is an ordered sample from $F(x) = x$, $x \in (0, 1)$ and N has the distribution given by* (1), *then:*

$$g(R) = \frac{2(1-R)\theta^2}{A_2(\theta)} \sum_{\substack{k \geqslant 2 \\ k \in T}} \binom{k}{2} a(k) (\theta R)^{k-2} \tag{7}$$

for $m \geqslant 1$,

$$E X_i^m = \frac{i}{A_i(\theta)} \sum_{\substack{k \geqslant i \\ k \in T}} \binom{k}{i} a(k) \theta^k \frac{(m+i-1)!\,(k-i)!}{(m+k)!} \tag{8}$$

and

$$E X_i X_j = \frac{i(j+1)}{A_j(\theta)} \sum_{\substack{k \geqslant j \\ k \in T}} \frac{a(k)\theta^k}{(k+1)(k+2)}, \quad \text{for } i < j. \tag{9}$$

Proof. (7) is a straightforward consequence of (6), while (8) and (9) we obtain from Lemma 1.

(iii) Particular cases. It is known that (1) with $T = \{0, 1, ..., n\}$, $a(k) = \binom{n}{k}$, $\bar{f}(\theta) = (1+\theta)^n$, $\theta = p/q$, where $0 < p < 1$, $p + q = 1$, reduces to the binomial distribution with parameters p and n.

If $T = \mathrm{N} \cup \{0\}$, $a(k) = (-1)^k \binom{-n}{k}$, $\bar{f}(\theta) = (1-\theta)^{-n}$, $\theta = q$, $0 < q < 1$, then (1) gives the negative binomial distribution with parameters q and n.

Putting $T = \mathrm{N} \cup \{0\}$, $a(k) = 1/k!$, $\bar{f}(\theta) = e^\theta$, $\theta = \lambda > 0$, we get the Poisson distribution with parameter λ.

Using the above facts one can get from (2) – (4) and from (6) – (9) the results of [6].

The above given considerations concerning the moments of the ordered statistics lead us to the following combinatorial formulae:

Corollary. *If T, θ, $\bar{f}(\theta)$ and $a(k)$ are quantities determining the binomial, the negative binomial and Poisson distribution, then for $m \geqslant 1$*

$$\sum_{\substack{k \geqslant i \\ k \in T}} \frac{k!}{(m+k-1)!} a(k)\theta^k \left[\frac{1}{\theta} - \frac{(k+1)}{m+k}\, \frac{a(k+1)}{a(k)}\right] = \frac{i!}{(m+i-1)!} a(i)\theta^{i-1},$$

i.e. explicite

$$\sum_{k=i}^{n} \frac{k!}{(m+k-1)!} \binom{n}{k} \left(\frac{p}{q}\right)^k \left[\left(\frac{p}{q}\right)^{-1} - \frac{m-k}{m+k}\right] = \frac{i!}{(m+i-1)!} \binom{n}{i} \left(\frac{p}{q}\right)^{i-1},$$

$$\sum_{k=i}^{\infty} \frac{k!}{(m+k-1)!} (-1)^k \binom{-n}{k} q^k \left[\frac{1}{q} - \frac{n+k}{m+k}\right] = \frac{i!}{(m+i-1)!} (-1)^i \binom{-n}{i} q^{i-1},$$

$$\sum_{k=i}^{\infty} \frac{1}{(m+k-1)!} \lambda^k \left[\frac{1}{\lambda} - \frac{1}{m+k}\right] = \frac{1}{(m+i-1)!} \lambda^{i-1}.$$

Proof. The above formulae we obtain from (8) and from the results of [6].

3. N has an inflated power series distribution.

(i) Distribution of order statistics.

A random variable N is said to have the inflated (at the point $k = l$, $l \in T$) power series distribution (IPSD), if the probability function of N is of the form

$$p(k;\ \theta, \alpha) = P[N = k] = \begin{cases} \beta + \alpha \dfrac{a(k)\,\theta^k}{\bar{f}(\theta)} & \text{for } k = l \\ \alpha \dfrac{a(k)\,\theta^k}{\bar{f}(\theta)} & \text{for } k \in T - l \end{cases} \tag{10}$$

where $0 < \alpha < 1$, $\alpha + \beta = 1$, and the symbols T, $a(k)$, $\bar{f}(\theta)$, θ are the same as in the definition of PSD.

It is obvious that in the case $\alpha = 1$ IPSD reduces to PSD. Putting $\gamma(\theta) = (\beta/\alpha)\bar{f}(\theta)$, we have

Lemma 3. *If $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_N$ is an ordered sample of size N, where N has IPSD* (10), *then:*

a) *the conditional* p.d.f. *of X_i, $1 \leqslant i \leqslant N$ conditioned on the event $[N \geqslant i]$ is*

$$g(x_i) = \begin{cases} \dfrac{D_i(\theta, F) + \gamma(\theta)\binom{l}{i}(1-F_i)^l}{A_i(\theta) + \gamma(\theta)}\, i F_i^{i-1} (1-F_i)^{-i} f_i & \text{for } i \leqslant l \\ \dfrac{D_i(\theta, F)}{A_i(\theta)}\, i F_i^{i-1} (1-F_i)^{-i} f_i & \text{for } i > l; \end{cases} \tag{11}$$

b) *if T is finite, then the conditional* p.d.f. *of X_{N-i+1}, conditioned on the event $[N \geqslant i]$ is*

$$g(x_{N-i+1}) = \begin{cases} \dfrac{E_i(\theta, F) + \gamma(\theta)\binom{l}{i} F^{l}_{N-i+1}}{A_l(\theta) + \gamma(\theta)}\; i(1 - F_{N-i+1})^{i-1} F^{-l}_{N-i+1} f_{N-i+1} & \text{for } i \leqslant l \\ \dfrac{E_i(\theta, F)}{A_l(\theta)}\; i(1 - F_{N-i+1})^{i-1} F^{-i}_{N-i+1} f_{N-i+1} & \text{for } i > l \end{cases} \tag{12}$$

c) *the joint conditional* p.d.f. *of* X_i *and* X_j, $1 \leqslant i < j \leqslant N$, *conditioned on the event* $[N \geqslant j]$ *is*

$$g(x_i, x_j) = \begin{cases} \dfrac{L_j(\theta, F) + \gamma(\theta)\, l(l-1)\binom{l-2}{i-1,\, j-i-1,\, l-j}(1-F_j)^l}{A_j(\theta) + \gamma(\theta)}\, F_i^{i-1}\, [F_j - \\ \quad - F_i]^{j-i-1}\, [1 - F_j]^{-j} f_i f_j & \text{for } j \leqslant l \\ \dfrac{L_j(\theta, F)}{A_j(\theta)}\, F_i^{i-1}\, [F_j - F_i]^{j-i-1}\, [1 - F_j]^{-j} f_i f_j & \text{for } j > l \end{cases} \tag{13}$$

Proof. Consider the case when $i \leqslant l$. By (10), we have

$$P[N \geqslant i] = \sum_{\substack{k \geqslant i \\ k \in T}} P[N = k] = \sum_{\substack{i \leqslant k \leqslant l-1 \\ k \in T}} P[N = k] + P[N = l] + \sum_{\substack{k > l \\ k \in T}} P[N = k] =$$

$$= \frac{\alpha}{\overline{f}(\theta)}\, [A_l(\theta) + \gamma(\theta)]$$

Moreover, in this case, we have

$$\sum_{\substack{k \geqslant i \\ k \in T}} h(x_i \mid k) P[N = k] = \frac{\alpha}{\overline{f}(\theta)} \Big[\sum_{\substack{k \geqslant i \\ k \in T}} i\binom{k}{i} F_i^{i-1}\, [1 - F_i]^{k-i} f_i\, a(k)\, \theta^k +$$

$$+ \frac{\beta}{\alpha}\, \overline{f}(\theta)\, i\binom{l}{i} F_i^{i-1}\, (1 - F_l)^{l-i} f_i =$$

$$= \frac{\alpha}{\overline{f}(\theta)} \big[[D_l(\theta, F) + \gamma(\theta)\binom{l}{i}\, i(1 - F_l)^l] F_i^{i-1}\, i(1 - F_l)^{-i} f_i \big]$$

Hence we obtain the first part of (11).

Similar evaluations allows us to get for $i > l$:

$$P[N \geq i] = \frac{\alpha}{\bar{f}(\theta)} A_i(\theta)$$

and

$$\sum_{\substack{k \geq i \\ k \in T}} h(x_i \mid k) P[N = k] = \frac{\alpha}{\bar{f}(\theta)} i F_i^{i-1} (1 - F_i)^{-l} f_i D_i(\theta, F).$$

This leads us to the second part of (11). The formulae (12) and (13) can be obtained in an analogous way.

The p.d.f. of the smallest and largest (when T is finite) order statistics are directly obtained from (11) and (12) by putting $i = 1$.

Lemma 4. *Let R be the range of an ordered sample $X_1 \leq X_2 \leq \ldots \leq X_N$, where N is a random variable distributed according to* (10). *Then*

$$g(R) = \frac{2}{A_2(\theta) + \gamma(\theta)} \Big[\sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) \theta^k \int_{-\infty}^{\infty} [F(x + R) - F(x)]^{k-2} f(x) f(x + R) \, dx +$$

$$+ \gamma(\theta) \binom{l}{2} \int_{-\infty}^{\infty} [F(x + R) - F(x)]^{l-2} f(x) f(x + R) \, dx \Big]. \tag{14}$$

Proof. Since the p.d.f. of the range R is, for fixed k,

$$h(R) = k(k-1) \int_{-\infty}^{\infty} [F(x + R) - F(x)]^{k-2} f(x) f(x + R) \, dx,$$

then the p.d.f. of R for random N distributed according to (10) is given by (14).

(ii) A sample from a population uniformly distributed in (0, 1).

We now consider a sample $(X_1, X_2, \ldots, X_N)$ of size N from a population having a uniform distribution in (0, 1) i.e. $F(x) = x$ for $x \in (0; 1)$.

Theorem 2. *If $X_1 \leq X_2 \leq \ldots \leq X_N$ is an ordered sample from $F(x) = x$, $x \in (0, 1)$, and N has the distribution given by* (10), *then*

$$g(R) = \frac{2(1 - R)}{R^2 [A_2(\theta) + \gamma(\theta)]} \Big[\sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) (\theta R)^k + \gamma(\theta) \binom{l}{2} R^l \Big]; \tag{15}$$

for $m \geq 1$

$$E X_m^i = \begin{cases} \dfrac{i(m + i - 1)!}{A_i(\theta) + \gamma(\theta)} \Big[\sum\limits_{\substack{k \geq i \\ k \in T}} \binom{k}{i} a(k) \theta^k \dfrac{(k - i)!}{(m + k)!} + \gamma(\theta) \binom{l}{i} \dfrac{(l - i)!}{(m + l)!} \Big] & \text{for } i \leq l \\ \dfrac{i(m + i - 1)!}{A_i(\theta)} \sum\limits_{\substack{k \geq l \\ k \in T}} \binom{k}{i} a(k) \theta^k \dfrac{(k - i)!}{(m + k)!} & \text{for } i > l; \end{cases} \tag{16}$$

$$EX_iX_j = \begin{cases} \dfrac{i(j+1)}{A_j(\theta)+\gamma(\theta)} \Big[\sum\limits_{\substack{k \geq j \\ k \in T}} \dfrac{a(k)\,\theta^k}{(k+1)(k+2)} + \dfrac{\gamma(\theta)}{(l+1)(l+2)} \Big] & for\ j \leq l \\ \\ \dfrac{i(j+1)}{A_j(\theta)} \sum\limits_{\substack{k \geq j \\ k \in T}} \dfrac{a(k)\,\theta^k}{(k+1)(k+2)} & for\ j > l. \end{cases} \tag{17}$$

Proof. (15) is a straightforward consequence of (14), while (8) and (9) we obtain from Lemma 3.

(iii) Particular cases.

a) If the random variable N has the inflated binomial distribution with the parameters α, p, n, i.e. the probability function is of the form

$$P[N=k] = \begin{cases} \beta + \alpha \binom{n}{k} p^k q^{n-k} & \text{for } k = l \\ \\ \alpha \binom{n}{k} p^k q^{n-k} & \text{for } k = 0, 1, \ldots, l-1, l+1, \ldots, n, \end{cases}$$

where $0 < \alpha \leq 1$, $\alpha + \beta = 1$, $0 < p < 1$, $p + q = 1$, $n \in \mathbf{N}$, then the formulae (11) – (17) take the forms:

$$g(x_i) = \begin{cases} \dfrac{1}{B_i(n,p)+\gamma} \Big[np\binom{n-1}{i-1}(pF_i)^{i-1}(1-pF_i)^{n-i} f_i + \\ \quad + \gamma l \binom{l-1}{i-1} F_i^{l-1} (1-F_i)^{l-i} f_i & \text{for } i \leq l \\ \\ \dfrac{1}{B_i(n,p)} np\binom{n-1}{i-1}(pF_i)^{i-1}(1-pF_i)^{n-i} f_i & \text{for } i > l, \end{cases} \tag{11a}$$

where $\gamma = \beta/\alpha$ and $B_i(n,p) = \sum\limits_{k=i}^{n} \binom{n}{k} p^k q^{n-k}$;

$$g(x_{N-i+1}) = \begin{cases} \dfrac{1}{B_i(n,p)+\gamma} \Big[np^i \binom{n-1}{i-1}(1-F_{N-i+1})^{i-1}(q+pF_{N-i+1})^{n-i} f_{N-i+1} + \\ \quad + \gamma l \binom{l-1}{i-1}(1-F_{N-i+1})^{i-1} F_{N-i+1}^{l-i} f_{N-i+1} \Big] & \text{for } i \leq l \\ \\ \dfrac{1}{B_i(n,p)} np^i \binom{n-1}{i-1}(1-F_{N-i+1})^{i-1}(q+pF_{N-i+1})^{n-i} f_{N-i+1} & \text{for } i > l; \end{cases} \tag{12a}$$

$$g(x_i, x_j) = \begin{cases} \dfrac{1}{B_j(n,p)+\gamma}\,[n(n-1)p^j \binom{n-2}{i-1,\, j-i-1,\, n-j}(1-pF_j)^{n-j} + \\ \quad + \gamma l(l-1)\binom{l-2}{i-1,\, j-i-1,\, l-j}(1-F_j)^{l-j}]\, F_i^{i-1}[F_j - F_i]^{j-i-1} f_i f_j & \text{for } j \leq l \\ \dfrac{1}{B_j(n,p)}\, n(n-1)p^j F_i^{i-1}[F_j - F_i]^{j-i-1}\binom{n-2}{i-1,\, j-i-1,\, n-j} \\ \quad (1-pF_j)^{n-j} f_i f_j & \text{for } j > l; \end{cases} \tag{13a}$$

$$g(R) = \frac{1}{B_2(n,p)+\gamma}\Big\{n(n-1)p^2 \int_{-\infty}^{\infty} [p(F(x+R) - F(x)) + q]^{n-2} f(x) f(x+R)dx + \\ + \gamma l(l-1)\int_{-\infty}^{\infty} [F(x+R) - F(x)]^{l-2} f(x) f(x+R)\,dx\Big\}; \tag{14a}$$

$$g(R) = \frac{(1-R)(q+pR)^{-2}}{B_2(n,p)+\gamma}\,[n(n-1)p^2(q+pR)^n + \gamma l(l-1)(q+pR)^l]\,; \tag{15a}$$

$$EX_i^m = \begin{cases} \dfrac{m+i-1}{(n+m)p} EX_i^{m-1} + \dfrac{1}{(n+m)[B_l(n,p)+\gamma]}\Big[\gamma\dfrac{(n-l)(m+i-1)l}{p(m+l)} \cdot \\ \quad \cdot \binom{l-1}{i-1}\beta(m+i-1, l-i+1) - \binom{n-1}{i-1} np^{i-1} q^{n-i+1}\Big] & \text{for } i \leq l \\ \dfrac{m-i+1}{(n+m)p} EX_i^{m-1} - \dfrac{n}{(n+m)B_i(n,p)}\binom{n-1}{i-1} p^{i-1} q^{n-i+1} & \text{for } i > l, \end{cases} \tag{16a}$$

where $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$;

$$EX_iX_j = \begin{cases} \dfrac{i(j+1)}{B_j(n,p)+\gamma}\Big[\dfrac{B_{j+2}(n+2,p)}{p^2(n+1)(n+2)} + \dfrac{\gamma}{(l+1)(l+2)}\Big] & \text{for } j \leq l \\ \dfrac{i(j+2)\,B_{j+2}(n+2,p)}{(n+1)(n+2)\,B_j(n,p)} & \text{for } j > l. \end{cases} \tag{17a}$$

b) If the random variable N has the inflated negative binomial distribution with the parameters α, q, n, i.e. the probability function is of the form

$$P[N=k]=\begin{cases}\beta+\alpha\binom{-n}{k}p^n(-q)^k & \text{for } k=l\\ \alpha\binom{-n}{k}p^n(-q)^k & \text{for } k=0,1,\ldots,l-1,l+1,\ldots\end{cases}$$

where $0<\alpha\leqslant 1$, $\alpha+\beta=1$, $0<q<1$, $p+q=1$, $n\in\mathrm{N}$, then (putting $B_n^*(i,p)=\sum_{k=0}^{n}\binom{i}{k}p^k q^{i-k}$) the formulae (11) and (13) – (17) take the forms:

$$g(x_i)=\begin{cases}\dfrac{n\binom{n+i-1}{n-1}p^n q^i F_i^{i-1}f_i+\gamma l\binom{l-1}{i-1}(p+qF_l)^{n+1}[(p+qF_l)F_i]^{i-1}(1-F_l)^{l-i}f_i}{[B_{n-1}^*(n-i+1,p)+\gamma](p+qF_i)^{n+i}} & \text{for } i\leqslant l\\[2ex] \dfrac{n\binom{n+i-1}{n-1}p^n q^i F_i^{i-1}f_i}{B_{n-1}^*(n+i-1,p)(p+qF_i)^{n+i}} & \text{for } i>l;\end{cases}\tag{11b}$$

$$g(x_i,x_j)=\begin{cases}\dfrac{F_i^{i-1}[F_j-F_i]^{j-i-1}f_if_j}{[B_{n-1}^*(n+j-1,p)+\gamma](p+qF_j)^{n+j}}\Big[\binom{n+j-3}{i-1,\,j-i-1,\,n-1}(n+j-2)(n+j-1)p^nq^j+\gamma\binom{l-2}{i-1,\,j-i-1,\,l-j}(l-1)\,l\,(1-F_j)^{l-j}(p+qF_j)^{n+j}\Big] & \text{for } j\leqslant l\\[2ex] \dfrac{F_i^{i-1}[F_j-F_i]^{j-i-1}f_if_jp^nq^j}{B_{n-1}^*(n+j-1,p)(p+qF_j)^{n+j}}\binom{n+j-3}{i-1,\,j-i-1,\,n-1}(n+j-2)(n+j-1) & \text{for } j>l;\end{cases}\tag{13b}$$

$$g(R)=\frac{p^nq^2}{B_{n-1}^*(n+1,p)+\gamma}\Big[\int_{-\infty}^{\infty}\frac{f(x)f(x+R)}{[1-q(F(x+R)-F(x))]^{n+2}}\,dx+\gamma l(l-1)\int_{-\infty}^{\infty}[F(x+R)-F(x)]^{l-2}f(x)f(x+R)\,dx\Big];\tag{14b}$$

$$g(R) = \frac{p^n q^2 (1-R)}{B^*_{n-1}(n+1,p)+\gamma}\left[\frac{1}{(p+qR)^{n+2}} + \gamma l(l-1) R^{l-2}\right]; \qquad (15b)$$

for $m \geq 1$

$$EX_i^m = \begin{cases} \frac{p(m+i-1)}{q(n-m)} EX_i^{m-1} - \frac{1}{(n-m)[B^*_{n-1}(n+i-1,p)+\gamma]} \cdot \\ \cdot \left[\gamma \frac{(m-qn+pl)}{q(m+l)} \binom{l-1}{i-1} l(m+i-1)\beta(m+i-1,l-i+1) + \frac{p^n q^{i-1}}{\beta(n,i)}\right] & \text{for } i \leq l \\ \frac{p(m+i-1)}{q(n-m)} EX_i^{m-1} - \frac{p^n q^{i-1}}{(n-m)B^*_{n-1}(n+i-1,p)\beta(n,i)} & \text{for } i > l \end{cases} \qquad (16b)$$

$$EX_iX_j = \begin{cases} \frac{i(j+1)}{B^*_{n-1}(n+j-1,p)+\gamma}\left[\frac{p^2 I_q(j+2,n-2)}{q^2(n-2)(n-1)} + \frac{\gamma}{(l+1)(l+2)}\right] & \text{for } j \leq l \\ \frac{i(j+1)p^2 I_q(j+2,n-2)}{B^*_{n-1}(n+j-1,p)\, q^2(n-1)(n-2)} & \text{for } j > l \end{cases} \qquad (17b)$$

where

$$\beta(a,b) I_q(a,b) = \int_0^q t^{a-1}(1-t)^{b-1}\,dt$$

c) If the random variable N has the inflated Poisson distribution with the parameters α, λ, i.e. the probability function is of the form

$$P[N=k] = \begin{cases} \beta + \alpha \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = l \\ \alpha \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, .., l-1, l+1, ... \end{cases}$$

where $0 < \alpha \leq 1$, $\alpha + \beta = 1$, $\lambda > 0$ then the formulae (11) and (13) – (17) take the forms:

$$g(x_i) = \begin{cases} \dfrac{1}{P_l(\lambda)+\gamma}\left[\dfrac{\lambda(\lambda F_i)^{i-1} f_i}{(i-1)!} e^{-\lambda F_i} + \gamma l\binom{l-1}{i-1} F_i^{i-1}(1-F_i)^{l-i} f_i\right] & \text{for } i \leq l \\ \\ \dfrac{\lambda(\lambda F_i)^{i-1} f_i}{P_i(\lambda)(i-1)!} e^{-\lambda F_i} & \text{for } i > l, \end{cases} \tag{11c}$$

where $P_i(\lambda) = \sum\limits_{k=i}^{\infty} \dfrac{\lambda^k e^{-\lambda}}{k!}$,

$$g(x_i, x_j) = \begin{cases} \dfrac{1}{P_j(\lambda)+\gamma}\left[\dfrac{\lambda^2(\lambda F_i)^{i-1} e^{-\lambda F_j}[\lambda(F_j-F_i)]^{j-i-1} f_i f_j}{(i-1)!\,(j-i-1)!} + \right. \\ \left. + \gamma\binom{l-2}{i-1,\, j-i-1,\, l-j}(l-1)\, l F_i^{i-1}[F_j-F_i]^{j-i-1}[1-F_j]^{l-j} f_i f_j\right] & \text{for } j \leq l \\ \\ \dfrac{\lambda^2(\lambda F_i)^{i-1} e^{-\lambda F_j}[\lambda(F_j-F_i)]^{j-i-1} \cdot f_i f_j}{(i-1)!\,(j-i-1)!\,P_j(\lambda)} & \text{for } j > l, \end{cases} \tag{13c}$$

$$g(R) = \frac{1}{P_2(\lambda)+\gamma}\left[\lambda^2 e^{-\lambda} \int_{-\infty}^{\infty} e^{\lambda[F(x+R)-F(x)]} f(x) f(x+R)\, dx + \right.$$

$$\left. + \gamma l(l-1) \int_{-\infty}^{\infty} [F(x+R)-F(x)]^{l-2} f(x) f(x+R)\, dx\right] \tag{14c}$$

$$g(R) = \frac{1-R}{P_2(\lambda)+\gamma}\left[\lambda^2 e^{-\lambda(1-R)} + \gamma l(l-1) R^{l-2}\right]; \tag{15c}$$

for $m \geq 1$

$$EX_i^m = \begin{cases} \dfrac{m+i-1}{\lambda} EX_i^{m-1} - \dfrac{1}{P_i(\lambda)+\gamma}\left[\gamma \dfrac{(m+i-1)(m+l-\lambda) l}{\lambda(m+l)} \cdot \right. \\ \left. \cdot \binom{l-1}{i-1} \beta(m+i-1, l-i+1) + \dfrac{\lambda^{i-1}}{(i-1)!} e^{-\lambda}\right] & \text{for } i \leq l \\ \\ \dfrac{m+i-1}{\lambda} EX_i^{m-1} - \dfrac{\lambda^{i-1} e^{-\lambda}}{(i-1)!\, P_i(\lambda)} & \text{for } i > l, \end{cases} \tag{16c}$$

$$EX_iX_j = \begin{cases} \dfrac{i(j+1)}{P_j(\lambda)+\gamma}\left[\dfrac{P_{j+2}(\lambda)}{\lambda^2} + \dfrac{\gamma}{(l+1)(l+2)}\right] & \text{for } j \leq l \\ \\ \dfrac{i(j+1)}{\lambda^2 P_j(\lambda)} P_{j+2}(\lambda) & \text{for } j > l. \end{cases} \tag{17c}$$

Remark. The results of [6] can be obtained from (iii) by setting $\alpha = 1$.

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STRESZCZENIE

Niech $X_1 < X_2 < ... < X_N$ będą statystykami porządkowymi dla próby o liczebności N, gdzie N – zmienna losowa o wartościach całkowitych nieujemnych.

W pracy podaje się wzory na rozkłady i momenty statystyk porządkowych, w przypadku, gdy N ma rozkład typu PSD i IPSD.

W szczególności rozpatrzono przypadek, gdy X_1 ma rozkład jednostajny na (0, 1) a N – dwumianowy, ujemny dwumianowy i Poissona oraz powyższe rozkłady uogólnione.

РЕЗЮМЕ

Пусть $X_1 < X_2 < ... < X_N$ – порядковые статистики выборки объема N, где N – случайная величина, принимающая неотрицательные цели значения.

В работе дается формула для разпределений и моментов порядковых статистик в случае, когда N имеет распределение типа степенных рядов и обобщенных степенных рядов.

В частности рассмотрено случай, когда X_1 имеет равномерное распределение на (0, 1) и N – биномиальное, отрицательно-биномиальное и Пуассона распределения и эти же распределения обобщенные.