ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXII, 12 SECTIO A 1978

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Józef ZAJĄC

The Ahlfors class N and its connection with Teichmüller quasiconformal mappings of an annulus

Klasa N Ahlforsa i jej związek z odwzorcowaniami quasikonforemnymi Teichmüllera w pierścieniu

Класс N Альфорса и его связь с квазиконформными отображениями Тейхмюллера в круговом кольце

Ahlfors [1] investigated the class N of complex-valued L^{∞} functions ν in the unit disc for which the antilinear part of variation of quasiconformal mappings vanishes, where the mappings are generated by dilatation of the form $t\nu$, t being a real parameter. He gave two important characterizations of the class N.

Reich and Strobel proved (1968) that the class N contains functions of the form $\bar{\phi}/|\phi|$, where ϕ is holomorphic in the unit disc. The present author obtains analogues of these results in the case of annuli.

Introduction and preliminaires. Let μ be a complex – valued measurable function in an annulus $\Delta_r = \{z : r \leqslant |z| \leqslant 1\}$, $0 < r < 1$, which satisfies

$$\|\mu\|_\infty = \inf_{E} \sup_{z \in \Delta_r \setminus E} |\mu(z)| < 1$$

where the infimum is taken over all sets E with the plane measure zero.

It is well known that there exists exactly one number R, $0 < R < 1$, and one Q – quasiconformal mapping f of the annulus Δ_r onto Δ_R which satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z \tag{1}$$

with $f(1) = 1$, where $Q = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$.

Here by a normalized quasiconformal mapping we mean any quasiconformal mapping $f : \Delta_r \to \Delta_R$ which satisfies $f(1) = 1$.

Suppose that $\mu = \mu(t)$ depends analytically on a real parameter t when regarded as an element of $L^{\infty}(\Delta_r)$. It has been proved [1] that f depends analytically on t for every fixed z and also that $\partial/\partial t$ commutes almost everywhere with $\partial/\partial z$ and $\partial/\partial \bar{z}$.

We confine corselves to the case $\mu(t) = t\nu$, where $\|\nu\|_{\infty} < +\infty$, $0 \leqslant t < 1$, and denote explicitly the dependence of f on $\nu : f(z, t) = f[\nu](z, t)$, $r \leqslant |z| \leqslant 1$.

Let

$$f[\nu](z) = \lim_{t \to 0} 1/t\,[f[\nu](z, t) - z]. \tag{2}$$

This expression is well defined and depends linearly on ν [1]. From $f[\nu]_{\bar{z}} = t\nu f[\nu]_z$ it follows that

$$\dot{f}[\nu]_{\bar{z}} = \nu. \tag{3}$$

It is well known that (3) is satisfied only if

$$\dot{f}[\nu](\zeta) = 1/\pi \iint_{\Delta_r} \frac{\nu(z)}{\zeta - z}\,dxdy + F(\zeta) \tag{4}$$

with holomorphic F, where $\iint_{\Delta_r} = \lim_{\epsilon \to 0} \iint_{\Delta_r \setminus \Delta^{\epsilon}(\zeta)}$ and $\Delta^{\epsilon}(\zeta) = \{z : |\zeta - z| \leqslant \epsilon\}$.

Thus we have ([4], p. 33):

$$\dot{f}[\nu](\zeta) = \frac{\zeta}{2\pi} \iint_{\Delta_r} \sum_{k=-\infty}^{+\infty} \Big[\frac{\nu(z)}{z^2} \Big(\frac{\zeta + r^{2k} z}{\zeta - r^{2k} z} - \frac{1 + r^{2k} z}{1 - r^{2k} z} \Big) - \frac{\overline{\nu(z)}}{\bar{z}^2} \Big(\frac{1 + r^{2k} \zeta \bar{z}}{1 - r^{2k} \zeta \bar{z}} - \frac{1 + r^{2k} \bar{z}}{1 - r^{2k} \bar{z}} \Big) \Big]\, dxdy, \tag{5}$$

where the notation $\ldots + a_{-1} + a_0 + a_1 + \ldots$ is applied instead of $a_0 + (a_1 + a_1) + \ldots$ provided that the last series converges.

We see that $\dot{f}$ is a continuous linear operator which maps every $\nu \in L^{\infty}(\Delta_r)$ on a function $\dot{f}[\nu]$. In addition, the relations $|f[\nu](z, t)| = 1$ for $|z| = 1$ and $|f[\nu](z, t)| = R[\nu](t)$ for $|z| = r$ yield

$$\operatorname{Re}\{\bar{z}\dot{f}[\nu](z)\} = ½ \lim_{t \to 0} \frac{|f[\nu](z, t) - z|^2}{tzf[\nu](z, t)} = 0 \quad \text{for } |z| = 1 \tag{6}$$

and

$$\operatorname{Re}\{\bar{z}\dot{f}[\nu](z)\} = r \operatorname{Re} \lim_{t \to 0} \frac{r}{tz} \Big(\frac{f[\nu](z, t)}{R[\nu](t)} - \frac{z}{r} \Big) + \lim_{t \to 0} \frac{1}{t} \{R[\nu](t) - r\} = r\rho \quad \text{for } |z| = r,$$

where $\rho = \lim_{t \to 0} 1/t[R[\nu](t) - r]$. In analogy to the above we verify that

$$\operatorname{Re}[\bar{z}\dot{f}[i\nu](z)] = 0 \quad \text{for } |z| = 1 \tag{8}$$

and

$$\operatorname{Re}[\bar{z}\dot{f}[i\nu](z)] = r\rho^* \quad \text{for } |z| = r \tag{9}$$

where $\rho^* = \lim_{t \to 0} 1/t[R[i\nu](t) - r]$.

Finally we need the following result due to Ławrynowicz [3]:

$$\rho = \frac{r}{2\pi} \iint_{\Delta_r} \left[\frac{\nu(z)}{z^2} + \frac{\overline{\nu(z)}}{\bar{z}^2} \right] dx\,dy, \tag{10}$$

which yields

$$\rho^* = \frac{ir}{2\pi} \iint_{\Delta_r} \left[\frac{\nu(z)}{z^2} - \frac{\overline{\nu(z)}}{\bar{z}^2} \right] dx\,dy. \tag{11}$$

By a Teichmüller mapping of Δ_r onto Δ_R we mean any quasiconformal mapping f whose complex dilatation μ has a.e. (= almost everywhere) the form $t\bar{\phi}/|\phi|$, where $0 \leqslant t < 1$, and ϕ is a function meromorphic in $\operatorname{int}\Delta_r$, whose only singularities may be poles of the first order. By a normalized Teichmüller mapping we mean any Teichmüller mapping f which satisfies the condition $f(1) = 1$.

We prove first.

Theorem 1. *Let f be a teichmüller mapping in Δ_r. Then*

$$r \log r^2 \leqslant \rho \leqslant r \log r^{-2}, \tag{12}$$

where the equality is attained for $f_1(z) = e^{i\theta} z|z|^{-2t/(1+t)}$, $z \in \Delta_r$, on the left – hand side and for $f_2(z) = e^{i\theta} z^{2t/(1-t)}$, $z \in \Delta_r$, on the right – hand side, where θ is a real parameter.

Proof. From the geometric definition of quasiconformal mappings (cf. e.g. [2], p. 31) it follows that

$$\frac{1-t}{1+t} \leqslant \log R(t)/\log r \leqslant \frac{1+t}{1-t}$$

where $R(t) = R[\bar{\phi}/|\phi|](t)$.

Since $0 \leqslant r \leqslant 1$, then

$$\log r^{\frac{1+t}{1-t}} \leqslant \log R(t) \leqslant \log r^{\frac{1-t}{1+t}},$$

i.e.

$$\frac{r^{\frac{1+t}{1-t}} - r}{t} \leqslant \frac{R(t) - r}{t} \leqslant \frac{r^{\frac{1-t}{1+t}} - r}{t}.$$

But $\left.\frac{d}{dt} r^{\frac{1+t}{1-t}}\right|_{t=0} = -\left.\frac{d}{dt} r^{\frac{1-t}{1+t}}\right|_{t=0} = 2r\log r$. Therefore (12) follows.

1. The Ahlfors class N for an annulus Δ_r. Now we shall define the Ahlfors class N_r for an annulus Δ_r and study some properties of this class. To this end let us decompose the variation $\dot{f}[\nu]$ defined by (2) as follows:

$$(13) \qquad \dot{f}[\nu] = 1/2[\dot{f}[\nu] + i\dot{f}[i\nu]] + 1/2[\dot{f}[\nu] - i\dot{f}[i\nu]],$$

where the first part is antilinear and the second part is linear with respect to the complex multipliers. By the definition of $\dot{f}[\nu]$ and (3) we see that $[\dot{f}[\nu] + i\dot{f}[i\nu]]_{\bar{z}} = 0$ i.e.

$$(14) \qquad \Phi[\nu] = \dot{f}[\nu] + i\dot{f}[i\nu]$$

is always a holomorphic function. The antilinearity is expressed by $\Phi[i\nu] = -i\Phi[\nu]$. We denote by N_r the subspace of $L^\infty(\Delta_r)$ which is formed by all ν with $\Phi[\nu] = 0$. It is a complex linear subspace of $L^\infty(\Delta_r)$.

The following characterization of N_r is important.

Lemma 1. *An element ν of $L^\infty(\Delta_r)$ belongs to N_r if and only if $\dot{f}[\nu]$ satisfies the condition*

$$(15) \qquad \dot{f}[\nu](z) = \begin{cases} 0 & \text{for } |z| = 1 \\ \dfrac{z}{\pi} \displaystyle\iint_{\Delta_r} \frac{\nu(\zeta)}{\zeta^2}\, d\xi\, d\eta & \text{for } |z| = r \end{cases}$$

where $\zeta = \xi + i\eta$.

Proof. By (14) we have

$$\mathrm{Re}[\bar{z}\,\Phi[\nu](z)] = \mathrm{Re}[\bar{z}\dot{f}[\nu](z)] - \mathrm{Im}[\bar{z}\dot{f}[i\nu](z)],$$

and, analogously,

$$\operatorname{Im}[\bar{z}\,\Phi[\nu](z)] = \operatorname{Im}[\bar{z}\dot{f}[\nu](z)] + \operatorname{Re}[\bar{z}\dot{f}[i\nu](z)].$$

By (6) and (7) this yields

$$\operatorname{Re}[\![\bar{z}\phi[\nu](z)]\!] = \begin{cases} -\operatorname{Im}[\![\bar{z}\dot{f}[i\nu](z)]\!] & \text{for } |z| = 1, \\ r\rho - \operatorname{Im}[\![\bar{z}\dot{f}[i\nu](z)]\!] & \text{for } |z| = r, \end{cases}$$

and, by (9),

$$\operatorname{Im}[\![\bar{z}\phi[\nu](z)]\!] = \begin{cases} \operatorname{Im}[\![\bar{z}\dot{f}[\nu](z)]\!] & \text{for } |z| = 1, \\ r\rho^* + \operatorname{Im}[\![\bar{z}\dot{f}[\nu](z)]\!] & \text{for } |z| = r. \end{cases}$$

Hence

$$(16) \qquad \bar{z}\dot{f}[\nu](z) = \begin{cases} i\operatorname{Im}[\![\bar{z}\phi[\nu](z)]\!] & \text{for } |z| = 1, \\ r(\rho - i\rho^*) + i\operatorname{Im}[\![\bar{z}\phi[\nu](z)]\!] & \text{for } |z| = r. \end{cases}$$

Therefore $\Phi = 0$ implies (15) by virture of (10) and (11).

Conversely, if $\dot{f}[\nu]$ satisfies (15), we see that the function $z \to \bar{z}\,\Phi[\nu](z)$ has real values on $\partial\Delta_r$. Since $\bar{z}\Phi[\nu](z) = |z|^2 z^{-1}\,\Phi[\nu](z)$, then the holomorphic function $z \to z^{-1}\,\Phi[\nu]\cdot(z)$ has real values on $\partial\Delta_r$ as well and it is continuous on Δ_r. It can easily be seen that this function must be constant beign real in Δ_r. But $\Phi[\nu](1) = 0$, whence $\Phi[\nu](z) = 0$ in Δ_r. This completes the proof.

As an immediate consequence of this Lemma we obtain.

Theorem 2. *If ν belongs to N_r, then*

$$(17) \qquad \iint_{\Delta_r} \nu(z)\,g(z)\,dx\,dy = \frac{i}{2}\iint_{\Delta_r} \frac{\nu(\zeta)}{\zeta^2}\,d\xi\,d\eta \,.\int_{|z|=r} z\,g(z)\,dz$$

for all g holomorphic in $\operatorname{int}\Delta_r$ *with* $\iint_{\Delta_r} |g(z)|\,dx\,dy < +\infty$, *where* $\zeta = \xi + i\eta$.

Proof. Suppose that g is a holomorphic function in $\operatorname{int}\Delta_r$ with finite L^1-norm in Δ_r. Since $\|\nu\|_\infty < +\infty$ then, by (3), and by Green's formulae in the generalized form (see [2], p. 148) we have

$$\iint_{\Delta_r} \nu(z)\,g(z)\,dx\,dy = -i/2\int_{|z|=1} \dot{f}[\nu](z)\,g(z)\,dz + i/2\int_{|z|=r} \dot{f}[\nu](z)\,g(z)\,dz$$

which, by (15) yields (17) and this completes the proof. In the case $R = r$ we may show, using the technique of Ahlfors [1], that the condition (17) is also sufficient.

2. Relationship between the Ahlfors class N_r and Teichm]er mappings of an annulus. Recall that in the case of the unit disc the Ahlfors class N is defined like in the case of the annulus Δ_r (see [1]). Suppose that f is a Teichmüller mappinggenerated by the complex dilatation of the form $t\bar{\phi}/|\phi|$, where ϕ is holomorphic in Δ and $0 \leq t < 1$. If we assume in addition, that f keeps the boundary points fixed for a sequence of values t tending to zero, then $\bar{\phi}/|\phi| \in N$. This result has been obtained by Reich adn Strebel [5].

In the case of an annulus we are going to prove.

Theorem 3. *Suppose that:*

(i) *f is a Teichmüller mapping generated by the complex dilatation of the form $t\bar{\phi}/|\phi|$, where ϕ is holomorphic in Δ_r and $0 \leq t < 1$,*

(ii) *f maps Δ_r onto $\Delta_{R(t)}$ and satisfies: $f(e^{i\theta}, t) = e^{i\theta}$, $f(re^{i\theta}, t) = R(t)e^{i\theta(r,t)}$, $0 \leq t < 1$, $-\pi < \theta \leq \pi$ with $\theta(r, t) = \arg f(re^{i\theta}, t)$,*

(iii) *we have*

$$\theta_t(r, 0) = \frac{1}{2\pi i} \iint_{\Delta_r} \left[\frac{\overline{\phi(\zeta)}}{|\phi(\zeta)|} \frac{1}{\zeta^2} - \frac{\phi(\zeta)}{|\phi(\zeta)|} \frac{1}{\zeta^2}\right] d\xi d\eta \tag{18}$$

Then $\bar{\phi}/|\phi| \in N_r$.

Proof. Suppose that f is a mapping which satisfies the hypotheses of the theorem. Then f keeps the points of $[z : |z| = 1]$ fixed for every $0 \leq t < 1$ and we see that $\dot{f}[\bar{\phi}/|\phi|](z) = 0$ for $|z| = 1$, so we arrive at the first condition in (15). In order to verify the second condition in (15) let us note that

$$\dot{f}[\bar{\phi}/|\phi|](z) = z[\rho/r + i\theta_t(r, 0)] \quad \text{for } |z| = r \tag{19}$$

Therefore, by (10) and (18), we obtain

$$\dot{f}[\bar{\phi}/|\phi|](z) = z/\pi \iint_{\Delta_r} \frac{\overline{\phi(\zeta)}}{|\phi(\zeta)|} \frac{d\xi d\eta}{\zeta^2} \quad \text{for } |z| = r$$

By Lemma 1 this complete the proof.

3. Corollary. First we give.

Corollary 1. *If in Theorem 2 we additionaly assume that g has holomorphic extension to $[z : |z| \leq 1]$, then*

$$\iint_{\Delta_r} \nu(z) g(z)\, dx dy = 0 \tag{20}$$

Proof. By a well known Cauchy's theorem the relation-ship (17) reduces to (20) because

$$\int_{|z|=r} z \cdot g(z)\, dz = 0$$

and this suffices to conclude the proof.

The relationship mentioned in (20) is the so called relation of orthogonality in the weak sense (see[6], p. 5). It plays an important role in the parametrical method for quasiconformal mapping of the unit disc.

Corollary 2. *Let f be a Teichmüller mapping generated by the complex dilatation of the form $t\bar{\phi}/|\phi|$, where ϕ is holomorphic and $0 \leqslant t < 1$, which maps Δ_r onto $\Delta_{R(t)}$ and satisfies: $f(e^{i\theta}, t) = e^{i\theta}$, $f(re^{i\theta}; t) = R(t)e^{i\theta}$, $0 \leqslant t < 1, -\pi < \theta < \pi$. Then $\bar{\phi}/|\phi| \in N_r$.*

Proof. Suppose that f is a mapping which satisfies the hypotheses of the Corollary. By Theorem 3 we obtain the first condition in (15). The second condition in (15) is even simpler because in this case $\theta_t(r, 0) = 0$ and this, by Lemma 2, completes the proof.

As an immediate consequence of (5) we shall give.

Corollary 3. *An element ν of $L^\infty(\Delta_r)$ belongs to N_r if and only if*

$$(21) \qquad \dot{f}[\nu](\zeta) = \frac{\zeta}{2\pi} \iint_{\Delta_r} \sum_{k=-\infty}^{+\infty} \frac{\nu(z)}{z^2} \left(\frac{\zeta + r^{2k} z}{\zeta - r^{2k} z} - \frac{1 + r^{2k} z}{1 - r^{2k} z} \right) dxdy$$

for $r \leqslant |\zeta| \leqslant 1$.

These results have natural analogues in the case $r = 0$, i.e. for the mapping in Δ with an additional invariant point 0.

REFERENCES

[1] Ahlfors, L. V., *Some remarks on Teichmüller space of Riemann surface*, Ann. of Math. 74 (1961), 171–191.

[2] Lehto, O. and Virtanen, K. I., *Quasiconformal mappings in the plane*, Berlin–Heidelberg–New York 1973.

[3] Ławrynowicz, J., *On the parametrization of quasiconformal mappings with invariant boundary points in the unit disc*, Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys., 9 (1973), 739–741.

[4] Ławrynowicz, J., *On the parametrization of quasiconformal mappings in an annulus*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 18 (1964), 23–52.

[5] Reich, E. and Strebel, K., *On quasiconformal mappings which keep the boundary points fixed*, Trans. Amer. Math. Soc., 138 (1969), 211–222.

[6] Reich, E. und Strebel, K., *Einige Klassen Teichmüllerscher Abbildungen, die die Randpunkte festhalten*, Ann. Acad. Sci. Fenn. Ser. A I, 457 (1970), 19.

STRESZCZENIE

Autor rozważa klasę N_r Ahlforsa w pierścieniu, podaje jej charakteryzacje oraz związek z odwzorowaniami quasikonforemnymi Teichmüllera.

РЕЗЮМЕ

Автор рассматривает класс Nr Альфорса в круговом кольце, представляет его характеристику и связь с квазиконформными отображениями Тейхмюллера.