ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXII, 10 SECTIO A 1978

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Sequentially Iterative Processes and Applications to Volterra Fuctional Equations

Ciągowe procesy iteracyjne i ich zastosowanie do równań funkcyjnych Volterry

Последовательные итеративные процессы и их приложения к функциональным уравнениям типа Вольтерри

1. Throughout this note, we shall make use of the following set of general notations

i) $P(X) = [Y \subset X : Y \neq \emptyset]$, $F(X, Y) = [f : f : X \to Y]$
ii) $C(X, Y) = [f \in F(X, Y) : f$ is continuous$]$, $N = [0, 1, \dots]$, $N' = [1, 2, \dots]$
iii) $X^n = X \times X \times \dots \times X$ ('n' times) $\forall n \in N'$, $X^N = F(N, X)$
iv) T^n (resp. $T^{(n)}$) = the n-th iterate of $T \in F(X, X)$, $\forall n \in N$
v) $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$, $R_+^0 =]0, +\infty[$, $\bar{R}_+ = [0, +\infty]$
vi) $K = [f \in F(R_+, R_+) : f(0) = 0, f(t) < t, \forall t > 0]$
vii) $P = [f \in K : \lim_{n \to \infty} f^{(n)}(t) = 0, \forall t \geq 0]$
viii) $(R^n, \|\cdot\|)$ = the euclidean n-dimensional space with the euclidean norm $\|\cdot\|$, $\forall n \in N'$.

Definition 1.1. Let $T \in F(X^N, X)$. An (N, T)-iterative process is a sequence $[x(i, j) : i, j \in N] \subset X$, defined by

$$
\begin{aligned}
(1.1)\quad & (x(0,0), x(0,1), \dots) \in X^N \text{ is given} \\
& x(1,0) = T(x(0,0), x(0,1), \dots), \\
& x(1,j) = T(x(1,0), \dots, x(1,j-1), x(0,j), \dots) \\
& x(2,0) = T(x(1,0), x(1,1), \dots), \\
& x(2,j) = T(x(2,0), \dots, x(2,j-1), x(1,j), \dots) \\
& \dots\dots\dots\dots \quad \forall j \in N'.
\end{aligned}
$$

Definition 1.2. Let $T \in F(X^N, X)$. A point $z \in X$ is called a N-fixed point of T iff $z = T(z, \dots, z, \dots)$.

Now, for every $T \in F(X^N, X)$ let us define the so-called 'associated map' $T_N \in F(X^N, X^N)$ by the following recursive procedure

(1.2) $T_N(x)(0) = T(x(0), x(1), \ldots)$

$T_N(x)(1) = T(T_N(x)(0), x(1), \ldots)$

$T_N(x)(2) = T(T_N(x)(0), T_N(x)(1), x(2), \ldots)$

....

for every $x = (x(0), x(1), \ldots) \in X^N$.

Remark 1.1. Every (N, T)-iterative process $[x(i, j) : i, j \in N] \subset X$ defined by (1.1) is equivalent to an oridinary iterative process $[y_m : m \in N] \subset X^N$, $y_0 = (x(0, 0), x(0, 1), \ldots)$, $y_{m+1} = T_N(y_m)$, $\forall m \in N$, where T_N is the associated map.

Remark 1.2. If $z \in X$ is a N-fixed point of $T \in F(X^N, X)$ then $y = (z, \ldots, z, \ldots) \in X^N$ is a fixed point of the associated map $T_N \in F(X^N, X^N)$. Conversely, if $y = (y(0), y(1), \ldots) \in X^N$ is a fixed point of $T_N \in F(X^N, X^N)$ then, a) $y(0) = \ldots = y(n) = \ldots$, b) the element $z = y(0)$ $(= \ldots = y(n) = \ldots) \in X$ is a N-fixed point of $T \in F(X^N, X)$.

Now, let (X, d) be a generalized complete metric space (abbreviated g.c.m.s.) [2], and let $T \in F(X^N, X)$. The aim of this note is to give some sufficient conditions for the convergence of the $(N : T)$-iterative process (1.1) to a N-fixed point of T on one hand and an evaluation of this convergence, on the other hand. In 2. an auxiliary fixed point theorem is presented. This main result of this note is given in 3. It may be compared with those of Presic [4] and Taskovic [5].

Finally, in 4., the main result is applied to a certain class of Volterra functional equations, obtaining a partial extension of some results due to Capra [1] and Pomentale [3].

2. Let (X, d) be a given g.c.m.s. The following theorem is useful in this note.

Theorem 2.1. *Let $X_1 \in P(X), T \in F(X_1, X), f \in K$ be such that*

(2.1) X_1 *is d-closed;* $T(X_1) \subset X_1$; $f \in P$,

(2.2) $x, y \in X_1$, $\tau > 0$, $[d(x, y) \leq \tau] \Rightarrow [d(Tx, Ty) \leq f(\tau)]$,

(2.3) $X_1(T) = [x \in N_1 : d(x, Tx) < +\infty] \neq \emptyset$.

Then, there exist $S \in F(X_1(T), X_1)$, $\rho \in F(X_1(T), R_+)$, such that, for every element $x \in X_1(T)$

(2.4) $d(x, Sx) < +\infty$: *Sx is a fixed point of T,*

(2.5) *if $y \in X_1$, $d(x, y) < +\infty$ then,* a) $y \in X_1(T)$, b) $Sx = Sy$,

(2.6) $d(T^m x, Sx) \leq f^{(m)}(\tau)$, $\forall \epsilon$, $\tau \geq \rho(x)$, $\forall m \in N$.

Proof. Let $x \in X_1(T)$ be given. Put $\tau = d(x, Tx)$. By (2.2) $d(T^m x, T^{m+1} x) \leq f^{(m)}(\tau)$, $\forall m \in N$. As $f \in P$ this implies

(2.7) $\lim_{m\to\infty} d(T^m x, T^{m+1}x) = 0.$

Let $\epsilon > 0$. From (2.7) there exists $m(\epsilon) \in N$ such that $[m \geqslant m(\epsilon)]$ imply $[d(T^m x, T^{m+1}x) < \epsilon - f(\epsilon)] \leqslant \epsilon]$. Then (for a fixed $m \geqslant m(\epsilon)$), the formula (2.2) and the inequality $d(T^m x, T^{m+1}x) < \epsilon$ imply $d(T^{m+1}x, T^{m+2}x) \leqslant f(\epsilon)$ and therefore $d(T^m x, T^{m+2}x) \leqslant d(T^m x, T^{m+1}x) + d(T^{m+1}, T^{m+2}x) < \epsilon - f(\epsilon) + f(\epsilon) = \epsilon$; the formula (2.2) and the inequality $d(T^m x, T^{m+2}x) < \epsilon$ imply $d(T^{m+1}x, T^{m+3}x) \leqslant f(\epsilon)$ and hence $d(T^m x, T^{m+3}x) \leqslant d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+3}x) < \epsilon - f(\epsilon) + f(\epsilon) = \epsilon, \dots$. Therefore $d(T^m x, T^{m+p}x) < \epsilon$, $\forall m \geqslant m(\epsilon)$, $\forall p \in N$, which shows that $[T^m x : m \in N] \subset \subset X_1$ is a Cauchy sequence. Let us define

(2.8) $Sx = \lim_{m\to\infty} T^m x,$

(2.9) $\rho(x) = \operatorname{diam}[T^m x : m \in N].$

Clearly, (2.4) holds (since (2.2) implies $d(Tx, Ty) \leqslant d(x, y)$, $\forall x, y \in X_1$ and thus, a fortiori T is a continuous map).

Let $y \in X_1$ be such that $\tau = d(x, y) < +\infty$. From (2.2), $d(Tx, Ty) \leqslant f(\tau) < +\infty$, and this gives $d(y, Ty) \leqslant d(y, x) + d(x, Tx) + d(Tx, Ty) < +\infty$, i.e., $y \in X_1(T)$. On the other hand, again from (2.2), we have $d(T^m x, T^m y) \leqslant f^{(m)}(\tau)$, $\forall m \in N$, and so, $\lim_{m\to\infty} d(T^m x, T^m y) = 0$, proving (2.5). Finally, from the evident inequality $d(x, Sx) \leqslant \tau$, $\forall \tau$, $\tau \geqslant \rho(x)$ and (2.2), we get (2.6).

Remark 2.1. A different choice of the function ρ is $\rho(x) = d(x, Sx)$, $\forall x \in X_1(T)$. On the other hand, if X_1 is bounded, the a useful choice of the function ρ is $\rho(x) = \operatorname{diam}(X_1)$, $\forall x \in X_1(T) = X_1$.

3. Let (X, d) be a g.c.m.s. The main result of this note is the following.

Theorem 3.1. *Let* $X_1 \in P(X)$, $T \in F(X_1^N, X)$, $f \in K$ *be such that*

(3.1) X_1 *is d-closed;* $T(X_1^N) \subset X_1$; $f \in P$,

(3.2) $(x(0), x(1), \dots) \in X_1^N$, $(y(0), y(1), \dots) \in X_1^N$, $\tau > 0$, $d(x(i), y(i)) \leqslant \tau$ *for each* $i \in N$ *imply* $d(T(x(0), x(1), \dots), T(y(0), y(1), \dots)) \leqslant f(\tau)$,

(3.3) *the set* $X_1^N(T)$ *of all* $(x(0), x(1), \dots) \in X_1^N$ *with the property* $\max[d(x(i), y(i)) : i \in \in N] < +\infty$, *(where* $y(0) = T(x(0), x(1), \dots)$, $y(1) = T(y(0), x(1), \dots)$, $y(2) = = T(y(0), y(1), x(2), \dots)$ *is not empty.*

Then, there exist $S \in F(X_1^N(T), X_1)$, $\rho \in F(X_1^N(T), R_+)$ *such that, for every element* $(x(0), x(1), \dots) \in X_1^N(T)$

(3.4) $\max[d(x(i), S(x(0), x(1), \dots)) : i \in N] < +\infty,$

(3.5) $S(x(0), x(1), \dots)$ *is a N-fixed point of T,*
(3.6) *if* $(y(0), y(1), \dots) \in X_1^N$ *is such that* $\max[d(x(i), y(i)) : i \in N] < +\infty$, *then*
a) $S(x(0), x(1), \dots) = S(y(0), y(1), \dots)$
b) $(y(0), y(1), \dots) \in X_1^N(T)$,
(3.7) *the* $(N : T)$*-iterative process* $[x(i, j) : i, j \in N] \subset X_1$, *where* $x(0, i) = x(i), \forall i \in N$, *converges to* $S(x(0), x(1), \dots)$ *in the following sense* $d(x(i, j), S(x(0), x(1), \dots)) \leqslant \leqslant f^{(i)}(\tau), \forall i, j \in N, \forall \tau, \tau \geqslant \tau(x(0), x(1), \dots)$.

Proof. Let us define $d_N \in F((X^N)^2, \bar{R}_+)$ by

(3.8) $d_N(x, y) = \max[d(x(i), y(i)) : i \in N]$.

for every $x = (x(0), x(1), \dots) \in X^N$, $y = (y(0), y(1), \dots) \in X^N$.

It is simply to verify that a) d_N is a generalized metric on X^N, b) (X^N, d_N) is a g.c.m.s. On the other hand, let $T_N \in F(X_1^N, X^N)$ be the associated map, defined by (1.2). From (3.1)–(3.3) it is clear that

(3.1)' $T_N(X_1^N) \subset X_1^N$; X_1^N is d_N-closed,
(3.2)' $x, y \in X_1^N, \tau > 0, d_N(x, y) \leqslant \tau$ imply $d_N(T_N x, T_N y) \leqslant f(\tau)$
(3.3) $x = (x(0), x(1), \dots) \in X_1^N(T)$ if and only if $d_N(x, T_N x) < +\infty$.

Therefore, theorem 2.1 is applicable (with X, X_1, d, T, replaced by $X^N, X_1^N, d_N\ T_N$, respectively). Denote also by S_N and ρ_N the corresponding mapping given by the quoted result and let us put

(3.9) $S(x(0), x(1), \dots) = S_N(x)(0) (= \dots = S_N(x)(n) = \dots)$
(3.10) $\rho(x(0) x(1), \dots) = \rho_N(x)$

for every $x = (x(0), x(1), \dots) \in X_1^N(T)$. The conclusions (3.4)–(3.7) easily follow from theorem 2.1 and remark 1.2.

Remark 3.1. Let $(x(0), x(1), \dots) \in X_1^N(T)$ be given. The $(N : T)$-iterative sequence $[x(i, j) : i, j \in N] \subset X_1$ has also the properties

(3.11) if $h \in F(N, N)$ is a bijection, then the sequence $[x(i, h(i)) : i \in N] \subset X_1$ convegers to $S(x(0), x(1), \dots)$ and $d(x(i, h(i)), S(x(0), x(1), \dots)) \leqslant f^{(i)}(\tau), \forall i \in N, \forall \tau, \tau \geqslant \rho(x(0), x(1), \dots)$,
(3.12) diam $[x(i, j) : j \in N] \leqslant f^{(i)}(\tau), \forall i \in N, \forall \tau, \tau \geqslant \rho(x(0), x(1), \dots)$.

Indeed, (3.11) follows immediately from (3.7). Furthermore, from remark 1.1 and theorem 2.1, $d(x(i-1, j)\, x(i, j)) \leqslant d_N(y_{i-1}, y_i) \leqslant f^{(i-1)}(\tau), \forall \tau, \tau \geqslant \rho(x(0), x(1), \dots)$, $\forall j \in N, \forall i \in N'$, and so, from (3.2), we get $d(x(i, j), x(i, k)) \leqslant f(f^{(i-1)}(\tau) = f^{(i)}(\tau), \forall \tau, \tau \geqslant \rho(x(0), x(1), \dots), \forall i \in N', \forall j, k \in N$, which was to be proved.

Remark 3.2. The mapping ρ has also the property

(3.10)' $\rho(x(0), x(1), \dots) = \max[d(i, k), x(j, k)) : i, j, k \in N]$.

4. In this paragraph, the main result will be applied to a certain class of Volterra functional equations. Firstly, let us denote

i) $X = C(R_+, R^n)$, $Y = C(R_+, R_+)$, $\tilde{Y} = [f \in Y : f^{-1}(R_+^0)$ is dense in $R_+]$

ii) $J = [I \in P(R_+) : I = \text{interval}]$, $D = [(t, s) \in (R_+)^2 : s \leqslant t]$.

Let us define $\|\cdot\| \in F(X, Y)$ by

iii) $\|x\|(t) = \|x(t)\|$, $\forall t \in R_+$, $\forall x \in X$ and for every $g \in \tilde{Y}$ define $\|\cdot\|_g \in F(X, \bar{R}_+)$ by

iv) $$\|x\|_g = \begin{cases} \inf[\lambda \in R_+ : \|x\| \leqslant \lambda g], & \text{if } [\lambda \in R_+ : \|x\| \leqslant \lambda g] \neq \emptyset \\ +\infty & \text{if } [\lambda \in R_+ : \|x\| \leqslant \lambda g] = 0 \end{cases}$$

for every $x \in X$. It is simply to verify that $(X, \|\cdot\|_g)$ is a generalized Banach space (respectively, a g.c.m.s., by the standard construction of its metric).

Let us put, for every $g \in Y$

v) $X_g = [x \in X : \|x\|_g < +\infty]$.

Now, let $\phi \in F(R_+, J)$ be a given application. For the sake of simplicity we shall denote henceforward

vi) $\hat{t} = \phi(t)$, $\forall t \in R_+$; $\hat{R}_+ = \phi(R_+)$.

Let $X_1 \in P(X)$. Suppose we have construced a family of mappings $[k(t) : t \in R_+] \subset$ $\subset F(X^N \times \hat{R}_+, R^n)$ such that

(4.1) for every $(x(0), x(1), \ldots) \in X_1^N$, the map $(t, s) \to k(t)(x(0), x(1), \ldots; \hat{s})$ is in $C(D, R^n)$.

Besides, let $x^0 \in X$, $e \in Y$ be given. Consider the following Volterra functional equation with transformed argument

(4.2) $x(t) = x^0(t) + \int_0^t k(t)(x, \ldots, x, \ldots; \widehat{e(s)})\, ds$, $\forall t \in R_+$

or, in an abstract form, $x = T(x, \ldots, x, \ldots)$, where $T \in F(X_1^N, X)$ is given, for every $x =$ $= (x(0), x(1), \ldots) \in X_1^N$, by

(4.3) $T(x(0), x(1), \ldots)(t) = x^0(t) + \int_0^t k(t)(x(0), x(1), \ldots; \widehat{e(s)})\, ds$, $\forall t \in R_+$.

From theorem 3.1 we obtain the following result.

Theorem 4.1. *Suppose there exist $g \in \tilde{Y}$, $[a(t) : t \in R_+] \subset F(Y \times \hat{R}_+, R_+)$ and $f \in K$, such that.*

(4.4) *X_1 is $\|\cdot\|_g$-closed, $f \in P$*

(4.5) *$(x(0), x(1), \ldots) \in X_1^N$, $y \in X$, $y(t) = x^0(t) + \int_0^t k(t)(x(0), x(1), \ldots ; \hat{e(s)})\, ds$, for each $t \in R_+$ implies $y \in X_1$,*

(4.6) *$(t, s) \in D$, $h \in Y$, $(x(0), x(1), \ldots) \in X_1^N$, $(y(0), y(1), \ldots) \in X_1^N$, the inequality $\| x(i)(r) - y(i)(r) \| \leqslant h(r)$ for each $r \in R_+$, $i \in N$ imply $\| k(t)(x(0), x(1), \ldots ; \bar{s}) - - k(t)(y(0), y(1), \ldots ; \bar{s} \| \leqslant a(t)(h, \bar{s})$,*

(4.7) *$\forall h \in Y$ the map $(t, s) \to a(t)(h, \bar{s})$ is in $C(D, R_+)$*

(4.8) *$\int_0^t a(t)(g\tau, \hat{e(s)})\, ds \leqslant f(\tau) g(t)$, $\forall t \in R_+$, $\forall \tau > 0$,*

(4.9) *The set $(X_1^N)_0$ of all $(x(0), x(1), \ldots) \in X_1^N$ such that there exists $\mu > 0$ with the property $\| x(i)(t) - y(i)(t) \| \leqslant \mu g(t)$, $\forall t \in R_+$, $\forall i \in N$, where*

$$y(0)(t) = x^0(t) + \int_0^t k(t)(x(0), x(1), \ldots ; \hat{e(s)})\, ds,$$

$$y(1)(t) = x^0(t) + \int_0^t k(t)(y(0), x(1), \ldots ; \hat{e(s)})\, ds,$$

.. $\forall t \in R_+$, is not empty.

Then there exists $S \in F((X_1^N)_0, X_1)$ such that, for every element $(x(0), x(1), \ldots) \in (X_1^N)_0$

(4.10) $\max [\| x(i) - S(x(0), x(1), \ldots) \|_g : i \in N] < +\infty$

(4.11) *$S(x(0), x(1), \ldots)$ is the unique solution of* (4.2) *in the set* $[y \in X_1 : \max [\| x(i) - - y \|_g : i \in N] < +\infty$

(4.12) *the N-iterative process $[x(i, j) : i, j \in N] \subset X_1$, $x(0, i) = x(i)$, $\forall i \in N$,*

$$x(i+1, 0)(t) = x^0(t) + \int_0^t k(t)(x(i, 0), x(i, 1), \ldots ; \hat{e(s)})\, ds,$$

$$x(i+1, 1)(t) = x^0(t) + \int_0^t k(t)(x(i+1, 0), (x(i, 1), \ldots ; \hat{e(s)})\, ds,$$

.. $\forall t \in R_+$, $\forall i \in N$

converges to $S(x(0), x(1), \ldots)$ in the following way

$$\| x(i, j) - S(x(0), x(1), \ldots) \|_g \leqslant f^{(i)}(\tau), \forall i, j \in N,$$

for every $\tau \geqslant \rho = \sup [\| x(i, p) - x(j, p) \|_g : i, j\, p \in N]$.

Proof. Let $(x(0), x(1), \ldots) \in X_1^N$, $(y(0), x(1), \ldots) \in X_1^N$, $\tau > 0$ be such that $\| x(i) - - y(i) \|_g \leqslant \tau$ $\forall i \in N$. From (iv) it follows that $\| x(i)(r) - y(i)(r) \| \leqslant g(r)\tau$, $\forall r \in R_+$, $\forall i \in N$, and from (4.6)–(4.8) we get $\| T(x(0), x(1), \ldots)(t) - T(y(0), y(1), \ldots)(t) \| \leqslant$

$\leqslant \int_0^t \| k(t)(x(0), x(1), \ldots ; \hat{e(s)}) - k(t)(y(0), y(1), \ldots ; \hat{e(s)}) \| ds \leqslant \int_0^t a(t)(g\tau, \hat{e(s)})\, ds \leqslant$
$\leqslant f(\tau)g(t)$, $\forall t \in R_+$, and this implies $\| T(x(0), x(1), \ldots) - T(y(0), y(1), \ldots) \|_g \leqslant f(\tau)$, i.e., (3.2) holds. On the other hand, obviously, (4.4) and (4.5) imply (3.1), (4.9) imply (3.3). Therefore, theorem 3.1 is applicable.

Remark 4.1. Suppose that a) $\{k(t)\ t \in R_+\} = [k]$ (i.e., the family $[k(t); t \in R_+]$ is independent of t); b) $x^0(t) = x^0 \in R^n$, $\forall t \in R_+$. Then (4.2) is equivalent with the following Cauchy problem

(4.13) $x'(t) = k(x, \ldots, x, \ldots ; \hat{e(t)}) \quad \forall t \in R_+$

(4.14) $x(0) = x^0$.

In this case, theorem 4.1 gives a method of 'sequentially' successive approximation for the solution of (4.13) and (4.14). From this point of view, it may be compared with some results of [1], [3].

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STRESZCZENIE

W pracy podane jest pewne „ciągowe twierdzenie o punkcie stałym". Dotyczy ono istnienia i jednoznaczności punktu stałego oraz zbieżności ciągów iteracyjnych. Twierdzenie to zastosowano do pewnej klasy równań Volterry.

РЕЗЮМЕ

В данной работе представлено некоторую „непрерывную теорему о неподвижной точке". Это относится к реальности и однозначности постоянной точки, а также сходимости итерационных последовательностей. Эту теорему применено к теории некоторого класса уравнений Вольтерри с перемещенным аргументом.