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On the Stability of Solutions of Differential Equations with Random Retarded Argument

O stabilności rozwiązań równań różniczkowych z losowo opóźnionym argumentem

Об устойчивости решений дифференциальных уравнений со случайно запаздывающим аргументом

This paper is an attempt of an exetension of Repin's results (cf. [1]) relating to the stability of solutions of differential equations with retarded argument to the case of random retardations.

The stability in this case has been also studied in [3–5] whereas other problems concerning the differential equations with random retarded argument have been investigated in [6–8].

I. Let us consider:

– a probability space (Ω, A, P) and an arbitrary (finite or infinite) interval $T \subset R$,
– a function $f: T \times R^{nm} \times \Omega \to R^n$ continuous on $T \times R^{nm}$ for almost all $\omega \in \Omega$ and A-measurable for all $(t, x^{(1)}, \dots, x^{(m)}) \in T \times R^{nm}$, where $x^{(j)} \in R^n$, $j = 1, 2, \dots, m$,
– a non-negative number $\tau \in R$ and stochastic processes

$$T_t^j : T \times \Omega \to R, \; j = 1, 2, \dots, m$$

such that for almost all $\omega \in \Omega$ sample paths $T_t^j(\omega)$ of T_t^j are continuous on T and

$$0 \leqslant T_t^j(\omega) \leqslant \tau, \; t \in T, \tag{1}$$

– a number $t_0 \in T$ such that $t_0 - \tau \in T$, and a stochastic process

$$\phi_t : \langle t_0 - \tau, t_0 \rangle \times \Omega \to R^n$$

for which almost all sample paths $\phi_t(\omega)$ are continuous on $\langle t_0 - \tau, t_0 \rangle$.

Definition 1. We say that the stochastic process

$$X_t : [T \cap \langle t_0 - \tau, \infty)] \times \Omega \to R^n$$

is a sample solution of the differential equation

$$\frac{dX_t}{dt} = f(t, X_{t-T_t^1}, \dots, X_{t-T_t^m}, \omega) \tag{2}$$

with the initial condition ϕ_t iff for almost all $\omega \in \Omega$:
1° the sample path $X_t(\omega)$ of process X_t is continuous on $T \cap \langle t_0 - \tau, \infty)$,
2° $X_t(\omega) = \phi_t(\omega)$, $t \in \langle t_0 - \tau, t_0 \rangle$,
3° for every $t \in T \cap \langle t_0, \infty)$

$$X_t(\omega) = \phi_{t_0}(\omega) + \int_{t_0}^{t} f(s, X_{s-T_s^1(\omega)}(\omega), \dots, X_{s-T_s^m(\omega)}(\omega), \omega)\, ds.$$

Besides, we say that this sample solution is unique iff for almost all $\omega \in \Omega$ and every sample solution Y_t, $t \in T \cap \langle t_0 - \tau, \infty)$ of equation (2) with the initial condition ϕ_t we have:

$$X_t(\omega) = Y_t(\omega), \ t \in T \cap \langle t_0 - \tau, \infty).$$

Theorem 1. *Let $L(t, \omega) : T \to R$ be for almost all $\omega \in \Omega$ a continuous function such that*

$$\| f(t, x^{(1)}, \dots, x^{(m)}, \omega) - f(t, \bar{x}^{(1)}, \dots, \bar{x}^{(m)}, \omega) \|$$

$$\leq L(t, \omega) \sum_{j=1}^{m} \| x^{(j)} - \bar{x}^{(j)} \|,$$

$$(x^{(1)}, \dots, x^{(m)}), (\bar{x}^{(1)}, \dots, \bar{x}^{(m)}) \in R^{nm}, \ t \in T.$$

Then there exists a unique sample solution X_t, $t \in T \cap \langle t_0 - \tau, \infty)$ of the differential equation (2) with the initial condition ϕ_t.

Proof. We choose a set $\Omega^* \in A$ with $P(\Omega^*) = 1$ such that for any $\omega \in \Omega^*$:
– sample paths of f, ϕ_t and T_t^j, $j = 1, 2, \dots, m$ are continuous (on their domains, respectively),
– the condition (1) is satisfied, and
– there exists a function $L(t, \omega)$ such as in the assumption of the theorem.

Let I be any closed interval with $I \subset T \cap \langle t_0 - \tau, \infty)$ and $t_0 - \tau, t_0 \in I$.

Now we fix any $\omega \in \Omega^*$ and put

$$X_0(t, \omega) = \begin{cases} \xi_t(\omega), & t \in \langle t_0 - \tau, t_0 \rangle \\ \xi_{t_0}(\omega), & t \in I \cap \langle t_0, \infty) \end{cases}$$

$$X_k(t, \omega) = \begin{cases} \xi_t(\omega), & t \in \langle t_0 - \tau, t_0 \rangle \\ \xi_{t_0}(\omega) + \int_{t_0}^{t} f(s, X_{k-1}(s - T_s^1(\omega), \omega), \dots, X_{k-1}(s - T_s^m(\omega), \omega), \omega)\, ds, & t \in I \cap \langle t_0, \infty) \end{cases}$$

$k = 1, 2, \dots$

It may be shown by induction that for every $k = 0, 1, 2, \dots$ and $t \in I \cap \langle t_0, \infty)$ it holds:

$$\| X_{k+1}(t, \omega) - X_k(t, \omega) \| < M \,[m \sup_{s \in I} L(s, \omega)]^k \frac{(t - t_0)^{k+1}}{(k+1)!},$$

where

$$M = \sup_{s \in I \cap \langle t_0, \infty)} \| f(s, X_0(s - T_s^1(\omega), \omega), \dots, X_0(s - T_s^m(\omega), \omega), \omega) \|.$$

Then

$$\| X_{k+1}(t, \omega) - X_k(t, \omega) \| \leq M \left[m \sup_{s \in I} L(s, \omega) \right]^k \frac{(\max I - t_0)^{k+1}}{(k+1)!},$$

$t \in I, \quad k = 0, 1, 2, \dots$

Hence a limit $X(t, \omega)$, $t \in I$ of the sequence $X_k(t, \omega)$, $k = 0, 1, 2, \dots$ is a unique solution of the differential equation

$$\frac{dX_t(\omega)}{dt} = f(t, X_{t - T_t^1(\omega)}(\omega), \dots, X_{t - T_t^m(\omega)}(\omega), \omega)$$

with the initial condition $\phi_t(\omega)$.

For $\omega \in \Omega^*$, $t \in I$ and $k = 0, 1, 2, \dots$, we assume:

$$X_k(t, \omega) = 0, \quad X(t, \omega) = 0$$

We shall show by induction that for $k = 0, 1, 2, \ldots$ and every $t \in I$ the function $X_k(t, \omega)$ is A-measurable.

We choose an arbitrary $k = 1, 2, \ldots$, and assume that for every $t \in I$ the function $X_{k-1}(t, \omega)$ is A-measurable. It is enough to show that $X_k(t, \omega)$ is A-measurable for all $t \in I \cap \langle t_0, \infty)$. Let $X_{k-1}(t, \omega) = 0$ for $\omega \notin \Omega^*$, $t \in I$, $t \in \mathcal{R}$ and

$$X_{k-1}(t, \omega) = \begin{cases} \phi_{t_0 - \tau}(\omega), & t \in (-\infty, t_0 - \tau\rangle \\ X_{k-1}(\max I, \omega), & t \in \langle \max I, \infty) \end{cases}$$

for $\omega \in \Omega^*$.
In view of Lemma 1.2 in [2] (p. 12–13) there exists a function $Y(t, \omega) : \mathcal{R} \times \Omega \to \mathcal{R}^n$, $B \times A$-measurable (where B is the Borel σ-field of $\mathcal{R}$) such that for $(t, \omega) \in \mathcal{R} \times \Omega^*$

$$X_{k-1}(t, \omega) = Y(t, \omega).$$

Choose any $t \in I \cap \langle t_0, \infty)$. It follows from the above mentioned lemma that for any fixed $s \in \langle t_0, t\rangle$ there exists $B^{nm} \times A$-measurable function $g(s, x^{(1)}, \ldots, x^{(m)}, \omega)$: $: \mathcal{R}^{nm} \times \Omega \to \mathcal{R}^n$ such that

$$f(s, x^{(1)}, \ldots, x^{(m)}, \omega) = g(s, x^{(1)}, \ldots, x^{(m)}, \omega), \quad (x^{(1)}, \ldots, x^{(m)}, \omega) \in \mathcal{R}^{nm} \times \Omega^*$$

(B^{nm} denotes the Borel σ-field of $\mathcal{R}^{nm}$).
Thus we have

$$X_k(t, \omega) = \begin{cases} 0, & \omega \notin \Omega^* \\ \phi_{t_0}(\omega) + \int_{t_0}^{t} g(s, Y(s - T_s^1(\omega), \omega), \ldots, Y(s - T_s^m(\omega), \omega), \omega)\, ds, & \omega \in \Omega^*. \end{cases}$$

Since all functions in this formula are measurable with respect to suitable σ-fields and the integral is the ordinary Riemann's one, it can be checked that $X_k(t, \omega)$ is A-measurable. Finally, for every $t \in I$ the function $X(t, \omega)$ is measurable too. Hence there exists a unique sample solution $X_t = X(t, \omega)$, $t \in I$ of the differential equation (2) with the initial condition ϕ_t. This solution can be extended to the entire interval $T \cap \langle t_0 - \tau, \infty)$.

II. Now, let us assume that $T = \langle a, \infty)$, $a \in \mathcal{R}$ (or $T = \mathcal{R}$) and for almost all $\omega \in \Omega$ the function f satisfies the conditions:

$$\| f(t, x^{(1)}, \ldots, x^{(m)}, \omega) - f(t, \bar{x}^{(1)}, \ldots, \bar{x}^{(m)}, \omega) \|$$

$$\leq \mathcal{L} \sum_{j=1}^{m} \| x^{(j)} - \bar{x}^{(j)} \|,$$

$$(x^{(1)}, \dots, x^{(m)}), (\bar{x}^{(1)}, \dots, \bar{x}^{(m)}) \in \mathcal{R}^{nm}, \ t \in T, \ \mathcal{L} \in \mathcal{R},$$

and

$$f(t, \bar{0}, \dots, \bar{0}, \omega) = \bar{0}, \ t \in T, \ \bar{0} = (0, \dots, 0) \in \mathcal{R}^n.$$

Definition 2. We say that the trival solution of the differential equation (2) is uniformly asymptotically *W*-stable iff:

$$\bigwedge_{p \in (0,1)} \bigvee_{\beta > 0} \bigwedge_{\eta > 0} \bigvee_{\theta > 0} \bigwedge_{\phi_t} \left(E \left[\sup_{s \in \langle t_0 - \tau, t_0 \rangle} \| \phi_s(\omega) \| \right] < \beta \right.$$

$$\left. \Rightarrow P \left[\| X_t(\omega) \| < \eta, \ t > \theta + t_0 \right] > p \right).$$

Let us consider a differential equation

$$\frac{dY_t}{dt} = g(t, Y_{t-T_t^1}, \dots, Y_{t-T_t^m}(\omega)) \tag{3}$$

where the function g satisfies the same assumptions as the function f (but its Lipschitz constant have not to be equal to L).

Theorem 2. *Let the differential equation* (2) *satisfy the following condition:*

$$\bigvee_{\alpha > 0} \bigvee_{B > 0} \bigvee_{\delta_0 > 0} \bigwedge_{0 < \delta < \delta_0} \bigwedge_{\varphi_t} \bigvee_{\substack{\Omega^* \in A \\ P(\Omega^*) = 1}} \bigvee_{\omega \in \Omega^*} \left[\sup_{s \in \langle t_0 - \tau, t_0 \rangle} \| \varphi_s(\omega) \| < \delta \Rightarrow (\| X_t(\omega) \| < \right. \tag{4}$$

$$\left. < B\delta e^{-\alpha(t - t_0)}, \ t \geq t_0) \right],$$

where the set Ω^* have not to be the same for different sample solutions of this equation with the initial condition ϕ_t. If there exists $\sigma > 0$ such small that

$$\frac{\sigma B}{L + \sigma} (e^{m(L + \sigma)(1/\alpha \ln 4B + \tau)} - 1) < ¼,$$

and such that

$$\bigvee_{\substack{\tilde{\Omega}\in A \\ P(\tilde{\Omega})=1}} \bigvee_{h>0} \bigwedge_{\omega\in\tilde{\Omega}} \bigwedge_{t\in T} \Big(\sup_{j=1,2,\dots,m} \|x^{(j)}\| < h \Rightarrow \tag{5}$$

$$\Rightarrow \|f(t,x^{(1)},\dots,x^{(m)},\omega)-g(t,x^{(1)},\dots,x^{(m)},\omega)\| \leqslant \sigma \sum_{j=1}^{m} \|x^{(j)}\|\Big)$$

then the trival solution of the differential equation (3) *is uniformly asymptotically W-stable.*

Proof. We choose α, B, δ_0, σ, $\tilde{\Omega}$, and h according to the assumptions of the theorem. Let $\epsilon=\min[2B\delta_0, h]$ and $\beta=\epsilon/2B$. For any $\eta>0$ we assume that $\theta=r(1/\alpha\ln 4B+\tau)$, where $r\in N$ and $\epsilon/2^r<\eta$.

Now we take any initial condition ϕ_t and a sample solution Y_t, $t\in T\cap\langle t_0-\tau,\infty)$ of the differential equation (3) with this initial condition. Let $\Omega^Y\in A$, $P(\Omega^Y)=1$ be a set such that for every $\omega\in\Omega^Y$ the sample path $Y_t(\omega)$, $t\in T\cap\langle t_0-\tau,\infty)$ of the stochastic process Y_t is a solution of the differential equation

$$\frac{dY_t(\omega)}{dt}=g(t, Y_{t-T_t^1(\omega)}(\omega),\dots, Y_{t-T_t^m(\omega)}(\omega),\omega)$$

with the initial condition $\phi_t(\omega)$.

Next, let $\Omega^D\in A$, $P(\Omega^D)=1$ be a set such that for any $\omega\in\Omega^D$:

– the sample paths of f, g, ϕ_t and T_t^j, $j=1,2,\dots,m$ are continuous (on their domains, respectively),

– the condition (1) is satisfied, and

– the sample paths of f and g are lipschitzean (the function f with the constant L).

We take any sample solution X_t, $t\in T\cap\langle t_0-\tau,\infty)$ of the differential equation (2) with the initial condition ϕ_t and find for it the set Ω^* according to (4). Let Ω^X be a set defined for the function X_t in the same manner as the set Ω^Y has been defined for Y_t.

Now we fix an arbitrary $\omega\in\Omega^{(0)}=\tilde{\Omega}\cap\Omega^Y\cap\Omega^D\cap\Omega^*\cap\Omega^X$ and assume that $\sup_{s\in\langle t_0-\tau,t_0\rangle}\|\phi_s(\omega)\|<\bar{\beta}$. Then (see the proof of Theorem 1 in [1]) the function $Y_t(\omega)$ satisfied following conditions:

$$\|Y_t(\omega)\|<\epsilon,\ t\in\langle t_0, t_0+1/\alpha\ \ln\ 4B+\tau\rangle$$

and

$$\|Y_t(\omega)\|<\epsilon/4B,\ t\in\langle t_0+1/\alpha\ \ln\ 4B,\ t_0+1/\alpha\ \ln\ 4B+\tau\rangle.$$

We consider the stochastic process Y_t on the interval $\langle t_0+1/\alpha\ln 4B, t_0+1/\alpha\ln 4B+\tau\rangle$

as an initial condition and we take any sample solution X_t, $t \in T \cap \langle t_0 + 1/\alpha \ln 4B, \infty)$ of the differential equation (2) with this initial condition. For this solution we take sets Ω_1^* and Ω_1^X in a similar manner as the sets Ω^* and Ω^X were choosen. If $\omega \in \Omega^{(1)} = \Omega^{(0)} \cap \Omega_1^* \cap \Omega_1^X$ and $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$ then, similarly as above

$$\| Y_t(\omega) \| < \epsilon/2,\ t \in \langle t_0 + 1/\alpha \ln 4B + \tau,\ t_0 + 2(1/\alpha \ln 4B + \tau)\rangle$$

and

$$\| Y_t(\omega) \| < \epsilon/8B,\ t \in \langle t_0 + 2\cdot 1/\alpha \ln 4B + \tau,\ t_0 + 2(1/\alpha \ln 4B + \tau)\rangle.$$

In this way we obtain for every $i = 0, 1, 2, \ldots$ a set $\Omega^{(i)} \in A$ with $P(\Omega^{(i)}) = 1$ such that for $\omega \in \Omega^{(i)}$ if $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$ then

$$\| Y_t(\omega) \| < \epsilon/2^i,\ t \in \langle t_0 + i(1/\alpha \ln 4B + \tau),\ t_0 + (i+1)(1/\alpha \ln 4B + \tau)\rangle.$$

Let

$$\hat{\Omega} = \bigcap_{i=0}^{\infty} \Omega^{(i)}.$$

We choose any $\omega \in \hat{\Omega}$. If $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$ then $\| Y_t(\omega) \| < \epsilon/2^r < \eta$ for $t > \theta + t_0$.

Hence

$$(6) \quad \bigvee_{\bar{\beta} > 0} \bigwedge_{\eta > 0} \bigvee_{\theta > 0} \bigvee_{\substack{\hat{\Omega} \in A \\ P(\hat{\Omega}) = 1}} \bigwedge_{\varphi_t} \bigwedge_{\omega \in \hat{\Omega}} [\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \varphi_s(\omega) \| < \bar{\beta} \Rightarrow$$

$$\Rightarrow (\| Y_t(\omega) \| < \eta,\ t > \theta + t_0)],$$

where the set $\hat{\Omega}$ have again not to be the same for different sample solutions of the differential equation (3) with the initial condition ϕ_t.

Let $p \in (0,1)$. We choose $\bar{\beta}$ according to (6). Let $\beta \in R$ and $0 < \beta < (1-p)\bar{\beta}$. Now we fix any $\eta > 0$ and find θ fulfilling (6). Finally we choose an arbitrary initial condition ϕ_t and assume that

$$E[\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \|] < \beta.$$

In view of Chebyshev inequality we get

$$(7) \quad P\left[\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta} \right] \geqslant 1 - \frac{E\left[\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| \right]}{\bar{\beta}} > p.$$

We consider any sample solution Y_t, $t \in T \cap \langle t_0 - \tau, \infty)$ of the differential equation (3) with the initial condition ϕ_t. By (6) and (7), we have:

$$P[\, \| Y_t(\omega) \| < \eta,\ t > \theta + t_0 \,] > p,$$

with completes the proof.

Corollary. *If the differential equation* (2) *fulfils the condition* (4) *and for almost all* $\omega \in \Omega$ *it holds:*

$$\| f(t, x^{(1)}, \dots, x^{(m)}, \omega) - g(t, x^{(1)}, \dots, x^{(m)}, \omega) \|$$

$$\leqslant \sum_{j=1}^{m} \| x^{(j)} \| \, \Psi\Big(\sum_{j=1}^{m} \| x^{(j)} \| \Big), \ t \in T,$$

where $\psi : R \to R$ *and* $\lim_{x \to 0} \psi(x) = 0$, *then the trival solution of* (3) *is uniformly asymptotically W-stable.*

Proof. You only need to note that almost surely for any $\sigma > 0$ there exists h from the condition (5).

Let us consider a differential equation

$$\text{(8)} \qquad \frac{dZ_t}{dt} = f(t, Z_{t-S_t^1}, \dots, Z_{t-S_t^m}, \omega),$$

where the stochastic processes S_t^j, $j = 1, 2, \dots, m$ satisfies the same assumptions as the processes T_t^j.

Theorem 3. *Let the differential equation* (2) *satisfy the condition* (4). *If there exists* $\rho > 0$ *such small that*

$$\rho m L (1 + B)(e^{mL(1/\alpha \ln 4B + 2\tau)} - 1) < 1/4$$

and such that for almost all $\omega \in \Omega$

$$\text{(9)} \qquad \big| T_t^j(\omega) - S_t^j(\omega) \big| < \rho, \ t \in T, \ j = 1, 2, \dots, m,$$

then the trival solution of the differential equation (8) *is uniformly asymptotically W-stable.*

Proof. We choose α, B, δ_0, ρ according to the assumptions of the theorem. Let $\bar{\Omega} \in A$, $P(\bar{\Omega}) = 1$ be a set on which the condition (9) is satisfied. Let $\epsilon = 2B\delta_0$ and $\bar{\beta} = \epsilon / 2B \cdot e^{mL\tau}$. For any $\eta > 0$ we assume that $\theta = r\, 1/\alpha \ln 4B + (2r + 1)\tau$, where $r \in N$ and

$(\epsilon/2^r) < \eta$. Next, we take any initial condition ϕ_t and a sample solution Z_t, $t \in T \cap \langle t_0 - - \tau, \infty)$ of the differential equation (8) with this initial condition. Let sets Ω^Z and Ω^D be defindet similarly as the sets Ω^Y and Ω^D have been defindet in the proof of Theorem 2. We consider the stochastic process Z_t on the interval $\langle t_0, t_0 + \tau\rangle$ as an initial condition and we take any sample solution X_t, $t \in T \cap \langle t_0, \infty)$ of the differential equation (2) with this initial condition. For this solution we take sets Ω^* and Ω^X in like manner as in the proof the previous theorem.

Now we fix an arbitrary $\omega \in \Omega^{(0)} = \bar{\Omega} \cap \Omega^Z \cap \Omega^D \cap \Omega^* \cap \Omega^X$ and assume that $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$. Then (see the proof of Theorem 2 in [1]) the function $Z_t(\omega)$ fulfils the following conditions:

$$\| Z_t(\omega) \| < \epsilon, \quad t \in \langle t_0 + \tau, t_0 + 1/\alpha \ln 4B + 3\tau\rangle$$

and

$$\| Z_t(\omega) \| < \epsilon/4B, \quad t \in \langle t_0 + 1/\alpha \ln 4B + \tau, t_0 + 1/\alpha \ln 4B + 3\tau\rangle.$$

Next, we consider a sample solution X_t, $t \in T \cap \langle t_0 + 1/\alpha \ln 4B + 2\tau, \infty)$ of the differential equation (2) with the initial condition

$$Z_t, \quad t \in \langle t_0 + 1/\alpha \ln 4B + 2\tau, \ t_0 + 1/\alpha \ln 4B + 3\tau\rangle.$$

Analogically as in the proof of the previous theorem we obtain for every $i = 0, 1, 2, \ldots$ a set $\Omega^{(i)} \in A$ with $P(\Omega^{(i)}) = 1$ such that for $\omega \in \Omega^{(i)}$ if $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$ then

$$\| Z_t(\omega) \| < \epsilon/2^i, \quad t \in \langle t_0 + i\, 1/\alpha \ln 4B + (2i+1)\tau, t_0 + (i+1)\, 1/\alpha \ln 4B + (2i+3)\tau\rangle.$$

Let

$$\hat{\Omega} = \bigcap_{i=0}^{\infty} \Omega^{(i)}$$

We choose any $\omega \in \hat{\Omega}$. If $\sup_{s \in \langle t_0 - \tau, t_0\rangle} \| \phi_s(\omega) \| < \bar{\beta}$ then $\| Z_t(\omega) \| < \epsilon/2^r < \eta$ for $t > \theta + t_0$.

It proves that the condition (6) is also satisfied for the differential equation (8). Thus the trivial solution of this equation is uniformly asymptotically W-stable.

III. Now, let us assume that the function f and initial condition ϕ_t are unrandom:

$$f(t, x^{(1)}, \ldots, x^{(m)}, \omega) = f(t, x^{(1)}, \ldots, x^{(m)}), \ \omega \in \Omega,$$

$$(t, x^{(1)}, \ldots, x^{(m)}) \in T \times R^{nm},$$

$$\phi_t(\omega) = \phi(t), \ \omega \in \Omega, \ t \in \langle t_0 - \tau, t_0\rangle.$$

Furthermore, let us take functions

$$\tau_j : T \to R,\ j = 1, 2, \ldots, m$$

which are continuous on T and such that

$$0 \leqslant \tau_j(t) \leqslant \tau,\ t \in T.$$

Let us consider differential equation

$$\frac{dx(t)}{dt} = f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) \tag{10}$$

and

$$\frac{dX_t}{dt} = f(t, X_{t-T_t^1}, \ldots, X_{t-T_t^m}). \tag{11}$$

Definition 3. We say that the trival solution of the differential equation (10) is uniformly W-stable under persistent random retardations iff:

$$\bigwedge_{\epsilon>0} \bigwedge_{p\in(0,1)} \bigvee_{\delta>0} \bigvee_{\rho>0} \bigwedge_{\phi} \Big[\big(\sup_{s \in \langle t_0 - \tau,\, t_0\rangle} \|\phi(s)\| < \delta$$

$$\wedge\ E \Big[\sup_{j=1,2,\ldots,m} \ \sup_{s \in T \cap \langle t_0-\tau,\, \infty)} \big| \tau_j(s) - T_s^j(\omega) \big| \Big] < \rho \big)$$

$$\Rightarrow P \big[\|X_t(\omega)\| < \epsilon,\ \ t \geqslant t_0 \big] > p \Big].$$

Theorem 4. *If the trival solution of the differential equation* (10) *is uniformly asymptotically stable, i.e.*

$$\bigvee_{\beta>0} \bigwedge_{\eta>0} \bigvee_{\theta>0} \bigwedge_{\phi} \Big[\sup_{s \in \langle t_0-\tau,\, t_0\rangle} \|\phi(s)\| < \beta \tag{12}$$

$$\Rightarrow (\|x(t)\| < \eta,\ \ t > \theta + t_0) \Big]$$

then it is uniformly W-stable under persistent random retardations.

Proof. By Theorem 3 in [1], the trival solution of (10) is uniformly stable under persident disturbances of retardations, i. e.

$$\bigwedge_{\epsilon>0} \bigvee_{\delta>0} \bigvee_{\bar\rho>0} \bigwedge_{\varphi} [(\sup_{s\in\langle t_0-\tau, t_0\rangle} \| \varphi(s) \| < \delta \tag{13}$$

$$\bigwedge_{j=1,2,\ldots,m} \sup \sup_{s\in T_1\langle t_0-\tau, \infty)} | \tau_j(s) -$$

$$- \tau_j(s) | < \bar\rho) \Rightarrow (\| x^0(t) \| < \epsilon, t \geqslant t_0)].$$

where x^0 is solution of the following differential equation

$$\frac{dx^0(t)}{dt} = f(t, x^0(t-\tau_1^0(t)), \ldots, x^0(t-\tau_m^0(t)))$$

the equation fulfils the same conditions as (10).
For any choosen $\epsilon > 0$ we find δ and $\bar\rho$ such as in (13). Next, we fix any initial condition ϕ. Let a set $\bar\Omega$ be defindet similarly as the set $\Omega^D \cap \Omega^X$ in the proof of Theorem 2. By (13), for every $\omega \in \Omega$ if

$$\sup_{s\in\langle t_0-\tau, t_0\rangle} \| \phi(s) \| < \delta \text{ and } \sup_{j=1,2,\ldots,m} \sup_{s\in T\cap\langle t_0-\tau,\infty)} | \tau_j(s) - T_s^j(\omega) | < \bar\rho$$

then $\| X_t(\omega) \| < \epsilon$, $t \geqslant t_0$. So we have

$$\bigwedge_{\epsilon>0} \bigvee_{\delta>0} \bigvee_{\bar\rho>0} \bigwedge_{\varphi} \bigvee_{\substack{\bar\Omega\in A \\ P(\bar\Omega)=1}} \bigwedge_{\omega\in\bar\Omega} [(\sup_{s\in\langle t_0-\tau, t_0\rangle} \| \varphi(s) \| < \delta \wedge \tag{14}$$

$$\wedge \sup_{j=1,2,\ldots,m} \sup_{s\in T_1\langle t_0-\tau, \infty)} | \tau_j(s) - T_s^j(\omega) | < \bar\rho) \Rightarrow (\| X_t(\omega) \| < \epsilon, t \geqslant t_0)].$$

The end of the proof is analogous as previously.

Remark. The equations (2), (8) and the condition (6) can be considered instead of the equations (10), (11) and take condition (12) in Theorem 4. However the ihitial condition ϕ in this theorem should be urandom. If this initial condition is random then we can prove a condition similar to (14), but afterwards we ought to modify Definition 3 in a suitable manner.

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STRESZCZENIE

W pracy znajdują się warunki dostateczne stabilności rozwiązań równań różniczkowych z losowo opóźnionym argumentem, analogiczne do warunków podanych w pracy [1] dla równań nielosowych.

РЕЗЮМЕ

В работе находятся достаточные условия устойчивости решений дифференциальных уравнений со случайно запаздывающим аргументом, аналогичны условиям представленным в работе [1] для уравнений неслучайных.