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Strongly Starlike Functions of Higher Order

Funkcje mocno gwiaździste wyższego rzędu

Сильно звездные функции высшего порядка

1. Introduction. In [3], D. Brannan and W. Kirwan defined the class $S^*(a)$ of all function $f(z) = z + a_2z^2 + \ldots$ analytic in the unit disc U for which

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant a\pi/2 \qquad z \in U,\ a > 0. \tag{1.1}$$

(Note that (1.1) implies that $zf'(z)/f(z)$ is analytic and non-zero in U.) Functions in $S^*(a)$ are called strongly starlike of order a. The class $S^*(1)$ is the usual class of normalized univalent starlike functions and if $a < 1$, $S^*(a)$ consists only of bounded starlike functions [3]. The class $S^*(a)$, $0 < a \leqslant 1$, has been studied extensively (e.g. [1], [3], [6], [7], [8], [9]).

In this note we consider the case $a > 1$. We obtain sharp estimates on distortion and coefficients, using extreme point and subordination techniques, the function f_a defined by

$$f_a(z) = z\exp\Big(\int_0^z ((1+t)/(1-t))^a - 1\Big)dt/t \tag{1.2}$$

being essentially the only extremal function.

2. Basic properties of $S^*(a)$.

Theorem 2.1. *The extreme points of* $\{\log f(z)/z\colon f \in S^*(a)\}$, $a \geqslant 1$, *are precisely the functions* $\log f(x_a z)/xz$, $|x| = 1$.

Proof. As noted in [3], $f \in S^*(a)$ if and only if

$$zf'(z)/f(z) = (p(z))^a, \tag{2.1}$$

where $p(z)$ is subordinate to $(1+z)/(1-z)$. By [2, Theorem 2.1], the extreme points of $\{p^a\}$ are precisely the function of the form $((1+xz)/(1-xz))^a$ $|x| = 1$, for $a \geqslant 1$. The transformation

$$z(\log f(z)/z)' = (p(z))^a - 1$$

is linear and 1—1 from $\{\log f(z)/z\colon f\in S^*(a)\}$ onto $\{p^a\}$. The result now follows since extreme points of $\{\log f(z)/z\colon f\in S^*(a)\}$ are of the form

$$\log f(z)/z = \int_0^z \left[\left(\frac{1+xt}{1-xt}\right)^a - 1\right]\frac{dt}{t} = \int_0^{xz}\left[\left(\frac{1+y}{1-y}\right)^a - 1\right]\frac{dy}{y} = \log f_a(xz)/xz.$$

Corollary 2.2. *Let $f\in S^*(a)$, $a\geqslant 1$, then*

$$f_a(-r)\leqslant |f(re^{i\theta})|\leqslant f_a(r), \tag{2.2}$$

$$\big((1-r)/(1+r)\big)^a f_a'(-r)/r\leqslant |f'(re^{i\theta})|\leqslant \big(1+r/(1-r)\big)^a f_a(r)/r. \tag{2.3}$$

Proof. Inequality (2.2) follows upon exponentiation. To prove (2.3), note that by (2.1), if $z = re^{i\theta}$,

$$\big((1-r)/(1+r)\big)^a\leqslant |zf'(z)/f(z)|\leqslant \big((1+r)/(1-r)\big)^a.$$

Since $f_a(r) > r/(1-r)^2$ for $a>1$, f_a is not univalent in the unit disc U and thus the radius of univalence R_U of $S^*(a)$ is less than 1. The next three theorems give successively better lower bounds on R_U; an upper bound is obtained in Corollary 3.2. The exact determination of R_U appears quite difficult since $S^*(a)$ is not a linear invariant family. We note that the ideas of Theorems 2.3 and 2.4 are essentially due to Stankiewicz who proved analogous results if $a<1$.

Theorem 2.3. *If $f\in S^*(a)$ with $a\geqslant 1$, then f is convex for $|x|<r_c$, where*

$$r_c = a+1-(a^2+2a)^{1/2}.$$

The result is sharp with equality for $f=f_a$.

Proof. The proof given in [7] using a result of Causey and Merkes [4] is valid for all $a\geqslant 0$.

Theorem 2.4. *If $f\in S^*(a)$ with $a\geqslant 1$, then f is starlike for $|z|<r_s$, where*

$$r_s = \csc(\pi/2a)-\cot(\pi/2a).$$

The result is sharp, with equality for $f=f_a$.

Proof. Since $zf'(z)/f(z)$ is subordinate to $\big((1+z)/(1-z)\big)^a$,

$$|\arg zf'(z)/f(z)|\leqslant a\,|\arg(1+z)/1-z)|. \tag{2.4}$$

A short calculation yields, with $z = re^{i\theta}$,

$$\arg(1+z)/(1-z) = \arctan\big(2r\sin\theta/(1-r^2)\big). \tag{2.5}$$

Combining (2.4) and (2.5) we have

$$|\arg zf'(z)/f(z)|\leqslant \arctan(2r/1-r^2). \tag{2.6}$$

The result now follows since the left hand side of (2.6) is less than $\pi/2$ for $r < r_s$. Clearly equality holds in all these inequalities if and only if $f(z) = x^{-1}f_a(xz)$.

Theorem 2.5. *If $f \in S^*(a)$ with $a \geqslant 1$, then f is close-to-convex in $|z| < r_k$, where r_k is the radius of close-to-convexity of f_a.*

Proof. Following an idea of Krzyż [5], we will determine

$$\min \arg \frac{zf'(z)}{z_0 f'(z_0)}$$

where the minimum is taken over all $z = re^{i\theta}$ and $z_0 = re^{i\theta_0}$ with $|\theta| \leqslant \pi$, $|\theta_0| \leqslant \pi$.
It follows from (2.1) that

$$\arg \frac{zf'(z)}{z_0 f'(z_0)} = a \arg \frac{P(z)}{P(z_0)} + \arg \frac{f(z)}{f(z_0)} \tag{2.7}$$

$$= a \arg \frac{P(z)}{P(z_0)} + \log \frac{z}{z_0} + \operatorname{Im}\left[\int_0^z \left(P^a(t)-1\right)\frac{dt}{t} - \int_0^{z_0} \left(P^a(t)-1\right)\frac{dt}{t}\right]$$

Since $\{P^a\}$ is rotationally invariant, the minimum of (2.7) depends only on $\theta - \theta_0$. Let θ_0 be fixed. Now $\int_0^z \left(P^a(t)-1\right)dt/t$ is the limit of sums of the form

$$\sum_{k=1}^{n} \left(P^a\left(\frac{kr}{n}e^{i\theta}\right) - 1\right)/n.$$

Consequently, (2.7) is the limit of $\Phi_n\left(\log p^a(z)\right)$ where φ_n is entire. Since $P(z)$ is subordinate to $(1+z)/(1-z)$, $\Phi_n \log P^a(z))$ attains its minimum for each z only if $P(z) = (1+xz)/(1-xz)$, $|x| = 1$. Let z_n be chosen so this minimum is $\Phi_n \log P^a(z_n))$. If M is the minimum of (2.7), there is a subsequence $\Phi_m \log P^a(z_m))$ for which z_m converges to z', x_m converges to x' and hence (2.7) is minimized when $z = z'$ for the function $P(z) = (1+x'z)/(1-x'z)$. This completes the proof.

We note that it is possible to compute numerical values of r_k for specific a using (2.5).

9. Coefficient bounds. In [1], Brannan, Clunie and Kirwan studied the coefficient problem for $S^*(a)$ if $0 < a \leqslant 1$. They showed that

$$|a_2| \leqslant 2a \quad (0 < a \leqslant 1) \tag{3.1}$$

$$|a_3| \leqslant a \quad (0 < a < 1/3) \tag{3.2}$$

$$|a_3| \leqslant 3a^2 \quad (1/3 < a \leqslant 1) \tag{3.3}$$

$$|a_3| \leqslant 1/3 \quad a = 1/3. \tag{3.4}$$

The extremal functions for (3.1) and (3.3) are the functions $f(z) = z + 2az^2 + 3a^2z^3 + \ldots$ of (1.2) together with its rotations. Extremal functions for (3.2) and (3.4) are defined by

$$zf'(z)/f(z) = \left((1+xz^2)/(1-xz^2)\right)^a \quad |x| = 1$$

and

$$zf'(z)/f(z) = \lambda\left(\frac{1+xz}{1-xz}\right)^a + (1-\lambda)\left(\frac{1+xz^2}{1-xz^2}\right)^a, \quad |x| = 1,\ 0 \leqslant \lambda \leqslant 1.$$

In addition they showed that for each n, if a is sufficiently close to 1, $|a_n|$ is maximized by A_n, where

$$f_a(z) = z + \sum_{n=2}^{\infty} A_n z^n \tag{3.5}$$

We are able to solve completely the coefficient problem if $a \geqslant 1$.

Theorem 3.1. *Let* $f(z) = z + a_2z^2 + \ldots \in S^*(a)$, $a \geqslant 1$. *Then* $|a_n| \leqslant A_n$.

Proof. Let $(P(z))^a = 1 + b_1z + b_2z^2 + \ldots$ be the function defined by (2.1). By a result of Brannan, Clunie and Kirwan [2, Corollary 2.1], $|b_n| \leqslant B_n$, where

$$\left((1+z)/(1-z)\right)^a = 1 + B_1z + B_2z^2 + \ldots. \tag{3.6}$$

Comparing coefficients in (2.1) we obtain

$$(n-1)a_n = b_1a_{n-1} + b_2a_{n-2} + \ldots + b_n. \tag{3.7}$$

Since $a_2 = b_1$, the result is true if $n = 2$. Suppose that $|a_k| \leqslant A_k$, $2 \leqslant k \leqslant n-1$. Then from (3.6) and (3.7) we see that

$$(n-1)|a_n| \leqslant B_1A_{n-1} + B_2A_{n-2} + \ldots + B_n = A_n$$

This completes the proof.

Corollary 3.2. $R_U \leqslant 1/a$.

Proof. $A_2 = 2a$.

Theorem 3.3. *Let* $g(z) = z + a_{m+1}z^{m+1} + \ldots$ *be an m-fold symmetric function in* $S^*(a)$, $a \geqslant 1$. *Then*

$$|a_{mk+1}| \leqslant C_{mk+1} \quad k = 1, 2, \ldots$$

where $G(z) = z + C_{m+1}z^{m+1} + \ldots$ *is defined by*

$$= \frac{zG'(z)}{G(z)} = \left(\frac{1+z^m}{1-z^m}\right)^a.$$

Proof. The proof is analogous to that of Theorem 3.1, using the fact that if $Q(z)$ is an m-fold symmetric function with $Q(0) = 1$, $\mathrm{Re}\,Q(z) > 0$, then the coefficients of $(Q(z))^a$ are bounded by those of $\left(\frac{1+z^m}{1-z^m}\right)^a$.

REFERENCES

[1] Brannan, D., Clunie, J. and Kirwan, W., *Coefficient estimates for a class of starlike functions*, Canad. J. Math., 22 (1970), 476-485.

[2] ,, , *On the coefficient problem for functions of bounded boundary rotation*, Ann. Acad. Sci. Fenn. Ser. AI, 523 (1973), 18 pp.

[3] Brannan, D. and Kirwan, W., *On some classes of bounded univalent functions*, J. London Math. Soc., (2), 1 (1969), 431-443.

[4] Causey, W. and Merkes, E., *Radii of starlikeness of certain classes of analytic functions*, J. Math. Anal. Appl., 31 (1970), 201-204.

[5] Krzyż, J., *The radius of close-to-convexity within the family of univalent functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 20 (1962), 201-204.

[6] Leach, R., *On some classes of multivalent starlike functions*, Trans. Amer. Math. Soc., 209 (1975), 267-273.

[7] Stankiewicz, J., *Some extremal problems for the class S_a*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A., 25 (1971), 101-107.

[8] ,, , *Quelques problemes extremaux dans les classes des fonctions a-angulairement etoilees*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 20 (1966), 59-75.

[9] ,, , *Some remarks concerning starlike functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom, Phys., 23 (1970), 143-146.

STRESZCZENIE

W pracy autor bada tzw. funkcje mocno gwiaździste rzędu a, przy $a > 1$. Otrzymał on twierdzenia o zniekształceniu, oszacowanie współczynników, a także promień gwiaździstości i wypukłości dla funkcji rozważanej klasy.

РЕЗЮМЕ

В этой работе автор занимается так называемыми сильно звездообразными функциями порядка a, $a>1$. Получил он теоремы об искажению, оценку коэффициентов, а также радиус звездообразности и выпуклости для функции этого класса.