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On the Construction of some Measures of Noncompactness

O konstrukcji pewnych miar niezwartości

Об конструировании некоторых мер некомпактности

I. Introduction. Recently there have appeared a number of publications concerned with the notion of so-called measure of noncompactness. In the twenties K. Kuratowski ([8], [9]) introduced the function $\alpha(A)$. This function is defined on the set $\mathscr{M}$ of all bounded subsets of the metric space (X, ϱ) as follows:

$$\alpha(A) = \inf\{d > 0: A \text{ can be devided into a finite number of sets having diameters } < d\}.$$

K. Goebel [4] and L. S. Goldenstein, I.C. Gochberg, A. A. Markus ([5], [6]) have avoided the complications related to the count of the value of this function using the notion of the Hausdorff measure of noncompactness $\chi(A)$ $(\chi: \mathscr{M} \to \langle 0, \infty))$. Namely

$$\chi(A) = \inf\{\varepsilon > 0: A \text{ has a finite } \varepsilon\text{-net in } X\}.$$

Both these functions have the following properties ([3], [4], [11], [12])

1) $\mu(A) = 0 \Leftrightarrow A$ is precompact
2) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$
3) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$
3') $\mu(A \cup \{a\}) = \mu(A)$, $a \in X$
4) $\mu(\bar{A}) = \mu(A)$
5) Cantor's theorem: If A_i, $i = 1, 2, \ldots$ are closed and bounded sets in a complete metric space X such that $A_{i+1} \subset A_i$, $i = 1, 2, \ldots$ and $\lim\limits_{i\to\infty} \mu(A_i) = 0$, then the set $A_\infty = \bigcap\limits_{i=1}^{\infty} A_i$ is nonempty and compact.

In addition, when X is a linear space we have

6) $\mu(A+B) \leqslant \mu(A)+\mu(B)$
7) $\mu(a+A) = \mu(A)$, $a \in X$
8) $\mu(\lambda A) = |\lambda|\, \mu(A)$, $\lambda \in R$
9) $\mu(\operatorname{conv} A) = \mu(A)$,

where $\mu = a$ or $\mu = \chi$.

The following modification of the Hausdorff measure is familiar

$$\tilde{\chi}(A) = \inf\{\varepsilon > 0\colon A \text{ has a finite } \varepsilon\text{-net in } A\}$$

that satisfies only properties 1), 3′), 5) (see for example [2]).

In 1972 Istrăţescu [7] introduced the definition

$$\mathscr{I}(A) = \inf\{\varepsilon > 0\colon A \text{ contains no infinite } \varepsilon\text{-discrete set}\}$$

and as he remarked $\mathscr{I}$ has only properties 1), 3), 3′), 5). J. Daneš [2] (1974) wrote about other interesting properties of this function.

In a lot of papers ([2], [4], [6], [11]) we can find some exact formulas for the measure of noncompactness in a concrete metric spaces. A great deal of attention has been devoted to applications of the measure of noncompactness to the fixed point theory (see the references in [1]).

It seems that the natural approach to the notion of measure of noncompactness should be axiomatic. We can call the function $\mu\colon \mathscr{M} \to \langle 0, \infty)$ a measure of noncompactness if it satisfies some properties of type 1) – 9). Proper choice of axioms is of course the problem to discuss. In our opinion the axioms should be chosen in the way which guarantees usefulness of such functions in the fixed point theory and also in the way allowing to construct natural examples of such measures in concrete spaces.

Axiomatic definition can be found in Sadowski's paper [11]. A function $\psi\colon 2^E \supset \mathscr{M} \to \tilde{A}$, where E is a locally convex space, $\mathscr{M}$ denotes a set of all bounded subsets of E, $(\tilde{A}, \leqslant)$ is a partially ordered set, such that $\psi(\overline{\operatorname{conv}} A) = \psi(A)$, $A \in \mathscr{M}$ Sadowski called the measure of noncompactness. The function ψ does not satisfy, in general, the property 1), owing to it, ψ could be called the measure of noncompactness. But in the applications of to the fixed point theory Sadowski added some necessary conditions like 1) – 9).

In this paper we shall propose an axiomatic definition of a measure of noncompactness. We shall consider a Banach space with a general scheme of a criterion of compactness, from which we shall draw a formula for this measure. We shall prove that it satisfies all the axiomatic conditions. We shall compare it with the well-established measures in concrete Banach spaces.

II. Criterion of compactness and measure of noncompactness.

Let B be a Banach space and $\mathcal{M}$ the set of all bounded subsets of B. Assume, in addition, that there exists a nonzero sequence $(f_n)_{n\in N}$ of functionals defined on B, nonnegative, convex, lower semicontinuous and equibounded on every bounded subset of B. Assume, finally that there is in B a scheme of a criterion of compactness:

(S). *A* set $X \subset B$ is compact if and only if, when it is bounded, closed and the sequence $(f_n)_{n\in N}$ is uniformly convergent to 0 on X. It is easy to see that well-known criteria of compactness like Arzela, Riesz, Kolmogorov and the criterion of compactness in a Banach space with a basis can be written in the form of scheme (S) we shall consider this problem in the next section of our paper.

Let us introduce the following definition:

Def. 1. We say that the function $\mu\colon \mathcal{M}\to\langle 0, \infty)$ such that

a) $\mu(X) = 0 \Leftrightarrow X$ is precompact

b) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$

c) $\mu(X) = \mu(\bar{X})$

d) $\mu(\operatorname{conv} X) = \mu(X)$

e) $\mu[aX + (1-a)Y] \leqslant a\mu(X) + (1-a)\mu(Y)$, $0 \leqslant a \leqslant 1$

f) Cantor's theorem: If X_i, $i = 1, 2, \ldots$ are closed and bounded sets in B such that $X_{i+1} \subset X_i$, $i = 1, 2, \ldots$ and $\lim_{i\to\infty} \mu(X_i) = 0$ then the set $X_\infty = \bigcap_{i=1}^{\infty} X_i$ is nonempty and compact,

is the measure of noncompactness.

Now, let us consider the function μ giving by the following formula

$$(*) \qquad \mu(A) = \limsup_{n\to\infty} \sup_{x\in A} f_n(x), \quad A \in \mathcal{M}.$$

We shall show that μ has all properties a) — f), so it is a measure of noncompactness in the space B with the scheme (S). It is obvious that μ is well defined function on $\mathcal{M}$ with its values in $\langle 0, \infty)$.

Proof of a). Let X be a compact set in B. According to (S) X is bounded, closed and such that

$$\lim_{n\to\infty} \sup_{x\in X} f_n(x) = 0,$$

well now

$$\mu(X) = 0.$$

Inverse, if $\mu(X) = 0$ for closed X belongs to $\mathcal{M}$, then because $f_n \geqslant 0$, $n \in N$ we have for given $\varepsilon > 0$ there exists $n_0 \in N$ such that $\sup_{x\in X} f_n(x) < \varepsilon$ for $n \geqslant n_0$ and hence, by (S), X is compact.

Proof of b). It is trivial.

Proof of c). Let $X, \overline{X} \in \mathcal{M}$. Since $X \subset \overline{X}$ therefore, by b), $\mu(X) \leqslant \mu(\overline{X})$. Now, let $x \in \overline{X}$ i.e. $x = \lim_{k\to\infty} x_k$, $x_k \in X$, $k = 1, 2, \ldots$ Using the lower semicontinuity of f_n, $n = 1, 2, \ldots$ we have

$$f_n(x) \leqslant \liminf_{k\to\infty} f_n(x_k),\ n = 1, 2, \ldots$$

and obviously

$$f_n(x) \leqslant \limsup_{k\to\infty} f_n(x_k) \leqslant \sup_{z\in X} f_n(z),\ n = 1, 2, \ldots,$$

hence

$$\sup_{x\in\overline{X}} f_n(x) \leqslant \sup_{z\in X} f_n(z),$$

otherwise

$$\mu(\overline{X}) \leqslant \mu(X).$$

Proof of d). Let X, $\operatorname{conv} X \in \mathcal{M}$. It is easy to see that $\mu(X) \leqslant \mu(\operatorname{conv} X)$. It is also well-known, that $\sup_{x\in\operatorname{conv} X} f_n(x) \leqslant \sup_{x\in X} f_n(x)$, because of convexity of f_n, $n = 1, 2, \ldots$ therefrom

$$\mu(\operatorname{conv} X) \leqslant \mu(X).$$

Proof of e). Let X, Y, $aX + (1-a)Y \in \mathcal{M}$, $a \in \langle 0, 1\rangle$ and $x \in aX + (1-a)Y$ i.e. $x = ax_1 + (1-a)y_1$, $x_1 \in X$, $y_1 \in Y$, then

$$f_n(x) = f_n\big(ax_1 + (1-a)y_1\big) \leqslant af_n(x_1) + (1-a)f_n(y_1)$$
$$\leqslant a\sup_{x\in X} f_n(x) + (1-a)\sup_{y\in Y} f_n(y),$$

therefrom also

$$\sup_{x\in aX+(1-a)Y} f_n(x) \leqslant a\sup_{x\in X} f_n(x) + (1-a)\sup_{x\in Y} f_n(x).$$

This implies that

$$\mu(aX + (1-a)Y) \leqslant a\mu(X) + (1-a)\mu(Y),\ a \in \langle 0, 1\rangle.$$

Proof of f). Let $(x_i)_{i\in N}$ be a sequence such that $x_i \in X_i$, $i = 1, 2, \ldots$ and let us sign $X = \{x_1, x_2, \ldots\}$. Because obviously $\lim_{n\to\infty} f_n(x) = 0$, $x \in B$ hence

$$\mu(A) = \mu(A \setminus \{a\}),\ A \in \mathcal{M},\ a \in A.$$

Using this note we have

$$\mu(X) = \mu(\{x_1, x_2, \ldots\}) = \mu(\{x_2, x_3, \ldots\}) = \ldots = \mu(\{x_n, x_{n+1}, \ldots\})$$

and by b)

$$\mu(X) \leqslant \mu(X_n), \; n = 1, 2, \ldots,$$

but

$$\mu(X_n) < \varepsilon, \; n \geqslant n_0,$$

therefrom $\mu(X) = 0$ and X is precompact (by a)); so we can find in it a convergent sequence, which limit, obviously belongs to $\bigcap_{i=1}^{\infty} X_i$, hence X_∞ is nonempty. But because

$$X_\infty \subset X_i, \; i = 1, 2, \ldots,$$

we have

$$\mu(X_\infty) = 0.$$

By above, from the closedness of X_∞ and the property a) we obtain that X_∞ is compact.

III. Examples. A. Let us consider the space $C\langle a, b\rangle$ of continuos defined on $\langle a, b\rangle$ functions. According to the Arzela criterion of compactness [10] we define the sequence $(f_n)_{n\in N}$ as follows:

$$f_n(x) = \omega\left(x, \frac{1}{n}\right), \; x \in C\langle a, b\rangle,$$

where

$$\omega(x, \varepsilon) = \sup_{\substack{t,\bar{t}\in\langle a,b\rangle \\ |t-\bar{t}|<\varepsilon}} |x(t) - x(\bar{t})|$$

is the modulus of continuity of a function x. It is obvious that $f_n(x) \geqslant 0$, $x \in C\langle a, b\rangle$. The functionals are convex as well. Namely, let $x = \lambda x_1 + (1-\lambda) y_1$, where $x_1, y_1 \in C\langle a, b\rangle$ and $\lambda \in \langle 0, 1\rangle$ and let us take $t, \bar{t} \in \langle a, b\rangle$ such that $|t - \bar{t}| < \varepsilon$. Then

$$|x(t) - x(\bar{t})| \leqslant \lambda |x_1(t) - x_1(\bar{t})| + (1-\lambda)|y_1(t) - y_1(\bar{t})|$$
$$\leqslant \lambda \sup_{\substack{t,\bar{t}\in\langle a,b\rangle \\ |t-\bar{t}|<\varepsilon}} |x_1(t) - x_1(\bar{t})| + (1-\lambda) \sup_{\substack{t,\bar{t}\in\langle a,b\rangle \\ |t-\bar{t}|<\varepsilon}} |y_1(t) - y_1(\bar{t})|,$$

therefrom

$$\omega(x, \varepsilon) \leqslant \lambda\omega(x_1, \varepsilon) + (1-\lambda)\omega(y_1, \varepsilon).$$

It is easy to prove the continuity of f_n, $n = 1, 2, \ldots$. In fact let $\|x_n - x\| \to 0$ when $n \to \infty$, where $\|x\| = \sup_{t\in\langle a,b\rangle} |x(t)|$, $x \in C\langle a, b\rangle$.

Then for any $t, \bar{t} \in \langle a, b\rangle$ such that $|t - \bar{t}| < \varepsilon$ we have

$$|x_n(t) - x_n(\bar{t})| \leqslant |x_n(t) - x(t)| + |x(t) - x(\bar{t})| + |x(\bar{t}) - x_n(\bar{t})|$$
$$\leqslant 2\bar{\varepsilon} + |x(t) - x(\bar{t})|, \; n \geqslant n_0,$$

whence

$$\omega(x_n, \varepsilon) - \omega(x, \varepsilon) \leqslant 2\tilde{\varepsilon}, \ n \geqslant n_0.$$

In the same way we can prove that

$$\omega(x, \varepsilon) - \omega(x_n, \varepsilon) \leqslant 2\tilde{\varepsilon},$$

her efrom

$$|\omega(x_n, \varepsilon) - \omega(x, \varepsilon)| \leqslant 2\tilde{\varepsilon}, \ n \geqslant n_0.$$

This means that the modulus of continuity is the continuous function. Because

$$\omega\left(x, \frac{1}{n}\right) \leqslant \sup_{t\in\langle a,b\rangle} |x(t)| + \sup_{\bar{t}\in\langle a,b\rangle} |x(\bar{t})| = 2\|x\|,$$

so f_n, $n = 1, 2, \ldots$ are equibounded on every bounded set $X \subset C\langle a, b\rangle$. In consideration of all above things, the scheme (S) and the existence of $\lim\limits_{n\to\infty} \sup\limits_{x\in X} \omega\left(x, \frac{1}{n}\right)$ we can write

$$\mu(X) = \lim_{n\to\infty} \sup_{x\in X} \omega\left(x, \frac{1}{n}\right)$$

and as it is known [3] $\chi(X) = \frac{1}{2}\mu(X)$.

B. Now we consider the Banach space B with a basis $\{e_i\}_{i=1,2,\ldots}$. It is well-known that each $x \in B$ can be expressed in the following unique form

$$x = \sum_{i=1}^{\infty} a_i(x) e_i,$$

where $a_i(x)$ are so-called basic functionals.
Let us denote by R_n an operation

$$R_n x = \sum_{i=n}^{\infty} a_i(x) e_i$$

and because of the criterion of compactness in this space [10] let

$$f_n(x) = \|R_n x\|, \ x \in B.$$

The addition, continuity and the equiboundness of f_n are familiar, so f_n, $n = 1, 2, \ldots$ have needed properties.
Hence

$$\mu(X) = \lim_{n\to\infty} \sup_{x\in X} \|R_n x\|.$$

As K. Goebel [4] proved

$$\frac{1}{K}\limsup_{n\to\infty}\sup_{x\in X}\|R_n x\| \leqslant \chi(X) \leqslant \inf\{\sup_{x\in X}\|R_n x\|,\ n = 1, 2, \ldots\},$$

where $X \in \mathcal{M}$ and $K = \limsup_{n\to\infty}\|R_n\|$.

It is worth while to notice that when $K = 1$ we have $\chi(X) = \mu(X)$.

C. Let us consider the space $L^p\langle a, b\rangle$, $p > 1$.

a) Let $S_h x$ denotes a Stieklov's function for $x \in L^p\langle a, b\rangle$ [10]

$$(S_h x)(t) = \frac{1}{2h}\int_{t-h}^{t+h} x(s)\,ds,$$

where $x(t) = 0$ for $t \notin \langle a, b\rangle$.

According to the Kolmogorov criterion in $L^p\langle a, b\rangle$ we can define

$$f_n(x) = \|x - S_{1/n}x\|_{L^p},\ n = 1, 2, \ldots$$

It is easy to prove that f_n, $n = 1, 2, \ldots$ have needed properties, so

$$\mu(X) = \limsup_{n\to\infty}\sup_{x\in X}\|x - S_{1/n}x\|_{L^p},\ X \in \mathcal{M}.$$

In addition the following inequality is true [4]:

$$\chi(X) \leqslant \mu(X),\ X \in \mathcal{M},\ X \subset L^p\langle a, b\rangle.$$

b) There is the Riesz criterion of compactness in $L^p\langle a, b\rangle$ ([10], [13]), so we can take

$$f_n(x) = \|x - T_{1/n}x\|_{L^p},$$

where

$$(T_h x)(t) = x(t+h).$$

Here also $x(t) = 0$ for $t \notin \langle a, b\rangle$.

Therefore

$$\mu(X) = \limsup_{n\to\infty}\sup_{x\in X}\|x - T_{1/n}x\|_{L^p}.$$

IV. Remark. We can also consider the Banach space B with a criterion of compactness:

If a set $X \subset B$ is bounded and closed and a sequence $(f_n)_{n\in N}$ is uniformly convergent to 0 on X, then X is compact (where f_n, $n = 1, 2, \ldots$ are functionals defined in II).

It is only a sufficient condition. Then the function μ defined by (*) has obviously properties b) — f), but it is necessary to change a) as follows:

a') $\mu(X) = 0 \Rightarrow X$ is precompact.

REFERENCES

[1] Daneš, J., *On densifying and related mappings and their application in nonlinear functional analysis*, Theory of nonlinear operators, Akademie — Verlag, Berlin 1974, 15-56.

[2] Daneš, J., *On the Istrăţescu's measure of noncompactness*, Bull. Math. Soc. R. S. Roumanie, 16, 64, nr 4, (1972), 403-406.

[3] Darbo, G. *Punti uniti in trasformazioni a condominio non compatto*, Rend. Sem. Math. Univ. Padova, 24, (1955), 84-92.

[4] Goebel, K., *Grubość zbiorów w przestrzeniach metrycznych i jej zastosowania w teorii punktów stałych*, Rozprawa habilitacyjna, Uniw. M. Curie-Skłodowskiej, Lublin 1970.

[5] Goldenštein, L. S., Gochberg, I. C., Markus, A. S., *Исследование некоторых свойств линейных ограниченных операторов в связи с их q-нормой*, Уч. зап. Кишиневск. Ун-та, 29, (1957), 29-36.

[6] Goldenštein, L. S., Markus, A. S., *О мере некомпактности ограниченных множеств и линейных операторов*, Исслед. по алгебре и мат. анал., Кишинев, (1965), 45-54.

[7] Istrăţescu, V. I., *On a measure of noncompactness*, Bull. Math. Soc. R. S. Roumanie, 16, 64, nr. 2, (1972), 195-197.

[8] Kuratowski, K., *Topologie* t. I, II, Warszawa 1950, 1958.

[9] Kuratowski, K. *Sur les espaces complets*, Fund. Math. 15, (1930), 301-309.

[10] Lusternik, L. A., Sobolew, W. I., *Elementy analizy funkcjonalnej*, PWN Warszawa 1959.

[11] Sadowski, B. N., *Предельно компактные и уплотняющие операторы*, УМН, 27, (1972), 1, 81-146.

[12] Sadowski, B. N., *О мерах некомпактности и уплотняющих операторах*, Пробл. мат. анал. сложн. систем, вып. 2, Воронеж, (1968), 89-119.

[13] Yosida, K., *Functional analysis*, Springer — Verlag, Berlin 1965.

STRESZCZENIE

W pracy zajmujemy się związkiem między formułą na miarę niezwartości w przestrzeni metrycznej a działającym tam kryterium zwartości. Podajemy również dokładne formuły na miary niezwartości w przestrzeniach $C\langle a, b\rangle$, $L^p\langle a, b\rangle$ i w przestrzeni Banacha z bazą opierając się na znanych kryteriach zwartości.

РЕЗЮМЕ

В этой работе мы рассматриваем связь между формулой определяющей меру некомпактности в метрическом пространстве и действующим там критерием компактности. Мы также проводим точные формулы определяющие меру некомпактности в пространствах $C\langle a,b\rangle$, $L^p\langle a,b\rangle$ и в Ванаховом пространстве с базой, опираясь на известные критерия компактности.