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On the Location of Zeros of Polynomials

O rozmieszczeniu zer wielomianów

О расположении нулей полиномов

1. Introduction. The different results proved in this paper, though not having very much in common have been put together as they all deal with the location of zeros of polynomials. In Section 2 a classical result of Cauchy is considered for a special class of polynomials. In Section 3, extension of well-known Eneström-Kakeya theorem is considered for polynomials with complex coefficients. In Section 4, we obtain the minimum number of zeros of a polynomial, which the unit disc will contain provided the moduli of polynomial and its derivative satisfy certain conditions.

2. A classical result of Cauchy on the location of zeros of polynomial

$$p(z) = a_1 z^n + a_2 z^{n-1} + \ldots + a_n z + a_{n+1}$$

states that all the zeros are in the circle

$$|z| < 1 + A,$$

where

$$A = \operatorname*{Max}_{2 \leqslant j \leqslant n+1} \left| \frac{a_j}{a_1} \right|.$$

Let us consider the polynomial $p(z)$ with complex coefficients such that

$$|a_{j+1}| \leqslant |a_j|(1 + a/j), \; j = 1, 2, \ldots, n$$

for some non-negative real number a. We prove

Theorem 1. *Let* $p(z) = \sum_{k=1}^{n+1} a_k z^{n-k+1} (\not\equiv 0)$ *be a polynomial of degree n with complex coefficients such that*

$$|a_{j+1}| \leqslant |a_j|(1 + a/j), \; j = 1, 2, \ldots, n \tag{1}$$

for some non-negative real number a. Then p(z) has all its zeros in

$$|z| \leqslant \frac{2^{1/1+a}}{2^{1/1+a}-1}. \tag{2}$$

Proof of Theorem 1. From (1), it is easy to infer that

$$\left.\begin{aligned} |a_2| &\leqslant |a_1| \frac{(1+a)}{1!} \\ |a_3| &\leqslant |a_1| \frac{(1+a)(2+a)}{2!} \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ |a_{n+1}| &\leqslant |a_1| \frac{(1+a)\ldots(n+a)}{n!}. \end{aligned}\right\} \tag{3}$$

Now for $|z| > 1$, we have

$$|p(z)| \geqslant |a_1||z|^n - |a_2||z|^{n-1} - |a_3||z|^{n-2} \ldots - |a_n||z| - |a_{n+1}|$$

$$\geqslant |a_1||z|^n - |a_1|\frac{(1+a)}{1!}|z|^{n-1}| - |a_1|\frac{(1+a)(2+a)}{2!}|z|^{n-2} - \ldots$$

$$\ldots - |a_1|\frac{(1+a)\ldots(n+a)}{n!}$$

$$= |a_1||z|^n\left[1 - \sum_{k=1}^{n}\frac{(1+a)\ldots(k+a)1}{k!\,|z|^k}\right]$$

$$> |a_1||z|^n\left[1 - \sum_{k=1}^{\infty}\frac{(1+a)\ldots(k+a)1}{k!\,|z|^k}\right] = |a_1||z|^n\left[2 - \left(1 - \frac{1}{|z|}\right)^{-(1+a)}\right]. \tag{4}$$

Now the right hand side of inequality (4) will be greater than zero if

$$|z| > \frac{2^{1/1+a}}{2^{1/1+a}-1} \geqslant 1. \tag{5}$$

Hence on combining (4) and (5), we see that $p(z)$ has all its zeros in

$$|z| \leqslant \frac{2^{1/1+a}}{2^{1/1+a}-1}$$

and the theorem is proved.

3. The theorem of Eneström and Kakeya [5, p. 136] mentioned in the introduction states that if

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0$$

then the polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0$$

has all its zeros in the unit circle. If we do not assume the coefficients to be non-negative, the conclusion does not hold. However, we prove

Theorem 2. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k (\not\equiv 0)$ *be a polynomial of degree n with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2, \; k = 0, 1, \ldots, n$$

for some real β, *and*

$$|a_n| \geqslant a_{n-1}| \geqslant |a_{n-2}| \geqslant \ldots |a_2| \geqslant |a_1| \geqslant |a_0|,$$

then $p(z)$ *has all its zeros on or inside the circle*

$$|z| = 1 + \frac{2\sin\alpha}{|a_n|} \sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}. \tag{6}$$

For $\alpha = \beta = 0$, *this reduces to the Eneström-Kakeya theorem.*

Proof of Theorem 2. We may plainly assume $\beta = 0$. Let $\arg a_k = \alpha_k$, $\arg a_{k-1} = \alpha_{k-1}$. Then

$$|a_k - a_{k-1}|^2 = \big||a_k| e^{i\alpha_k} - |a_{k-1}| e^{i\alpha_{k-1}}\big|^2$$
$$= |a_k|^2 + |a_{k-1}|^2 - 2|a_k||a_{k-1}|\cos(\alpha_k - \alpha_{k-1})$$
$$\leqslant |a_k|^2 + |a_{k-1}|^2 - 2|a_k||a_{k-1}|\cos 2\alpha = (|a_k| - |a_{k-1}|)^2 + 4|a_k a_{k-1}|\sin^2\alpha.$$

Hence we have

$$|a_k - a_{k-1}| \leqslant (|a_k| - |a_{k-1}|) + 2|a_k a_{k-1}|^{\frac{1}{2}} \sin\alpha. \tag{7}$$

Now consider

$$g(z) = (1-z)p(z) = -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k + a_0$$

$$= -a_n z^{n+1} + P(z), \text{ say}. \tag{8}$$

For $|z| = 1$, we have

$$|P(z)| \leqslant \sum_{k=1}^{n} |a_k - a_{k-1}| + |a_0|$$

$$\leqslant \sum_{k=1}^{n} (|a_k| - |a_{k-1}|) + 2\Big(\sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}\Big)\sin\alpha + |a_0|, \text{ (by (7))}$$

$$= |a_n| + 2\Big(\sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}\Big)\sin\alpha.$$

Hence also

$$\left|z^n P\left(\frac{1}{z}\right)\right| \leqslant |a_n| + 2\left(\sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}\right) \sin\alpha \tag{9}$$

for $|z| = 1$. By the maximum modulus theorem (9) holds inside the unit circle as well. If $R > 1$ then $\frac{1}{R} e^{-i\theta}$ lies inside the unit circle for every real θ and from (9) it follows that

$$|P(Re^{i\theta})| \leqslant \left\{|a_n| + 2\left(\sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}\right) \sin\alpha\right\} R^n$$

for every $R \geqslant 1$ and θ real.

Thus for $|z| = R > 1$

$$|g(z)| = |-a_n z^{n+1} + P(z)| \geqslant |a_n| R^{n+1} - \left\{|a_n| + 2\left(\sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}\right) \sin\alpha\right\} R^n > 0$$

if

$$R > 1 + \frac{2\sin\alpha}{|a_n|} \sum_{k=1}^{n} |a_k a_{k-1}|^{\frac{1}{2}}.$$

From this the theorem follows.

It should be remarked here that an extension of Eneström-Kakeya theorem to polynomials with complex coefficients was obtained by Govil and Rahman [3, Theorem 2]. But in some cases, Theorem 2 gives better result than that obtained in case of Govil and Rahman, as the example

$$p(z) = 4z^5 + 3(\sqrt{.856} + .12i)z^4 + 2(\sqrt{.856} - .12i)z^3 + .2z^2 + 2(\sqrt{.99} - .1i)z + 1$$

shows.

4. A result of Ankeny and Rivlin [1, Theorem 2] states that if $p(z)$ is a polynomial of degree n with real coefficients having all zeros of non--positive real part such that for some $R > 1$

$$p(R) > p(1)\,\frac{R^k + R^n}{2},$$

k a non-negative integer, then $p(z)$ has at least $(k+1)$ zeros in $|z| < 1$.

We instead, consider a relation between $p'(1)$ and $p(1)$, and prove

Theorem 3. *If $p(z)$ is a polynomial with real coefficients having all the zeros with non-positive real part and if*

$$p'(1) > p(1)\,\frac{n+k}{2},$$

k a non-negative integer, then $p(z)$ has at least $(k+1)$ zeros in $|z| < 1$.

To prove the theorem, we need the following lemmas.

Lemma 1. *If $f(z)$ is a polynomial of degree n, then*

$$\operatorname*{Max}_{|z|=1} |f'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |f(z)|.$$

This lemma is well known Bernstein Theorem [2]

Lemma 2. *If $g(z)$ is a polynomial of degree n such that it has no zeros in $|z| < 1$, then*

$$\operatorname*{Max}_{|z|=1} |f'(z)| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |f(z)|.$$

This lemma is due to Lax [4].

Proof of Theorem 3. Suppose $p(z)$ has m zeros in $|z| < 1$ and $m \leqslant k$. Let

$$p(z) = (z-z_1)\ldots(z-z_m)(z-z_{m+1})\ldots(z-z_n)$$

and suppose $|z_j| < 1$ $(j = 1, 2, \ldots, m)$. Put

$$g(z) = (z-z_1)\ldots(z-z_m)$$

and

$$h(z) = (z-z_{m+1})\ldots(z-z_n).$$

The polynomials $p(z)$, $g(z)$ and $h(z)$ have positive coefficients. Hence

$$g'(1) \leqslant g(1)m \tag{10}$$

and

$$h'(1) \leqslant h(1)\frac{n-m}{2} \tag{11}$$

according to Lemmas 1 and 2 respectively.
Thus

$$\frac{p'(1)}{p(1)} = \frac{h'(1)}{h(1)} + \frac{g'(1)}{g(1)} \leqslant \frac{n-m}{2} + m \text{ (by (10) and (11))}$$

$$= \frac{n+m}{2} \leqslant \frac{n+k}{2}$$

giving thereby

$$p'(1) \leqslant p(1)\frac{n+k}{2}$$

a contradiction establishing the theorem.

We may apply Theorem 3 to the polynomial $z^n p(\overline{1/z})$ to get the following.

Corollary 1. *If $p(z)$ is a polynomial with real coefficients and having all the zeros with non-positive real part and if*

$$q'(1) > q(1)[n-\tfrac{1}{2}]$$

then $p(z)$ has no zeros in $|z| \leqslant 1$. Here $q(z)$ denotes the polynomial $z^n p(\overline{1/z})$.

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STRESZCZENIE

W pracy zawarte są pewne rezultaty dotyczące rozmieszczenia zer wielomianów. Pierwszy dotyczy ograniczenia modułu zer wielomianu $p(z) = \sum_{k=1}^{n+1} a_k z^{n-k+1}$, gdy jego współczynniki spełniają nierówności $|a_{j+1}| \leq (1+a/j)|a_j|$. Dalej udowodniono, że jeżeli współczynniki wielomianu $p(z) = \sum_{k=0}^{n} a_k z^k$ spełniają warunki

$$|\arg a_k - \beta| \leq a \leq \pi/2$$

$$|a_0| \leq |a_1| \leq \ldots \leq |a_n|$$

to wszystkie zera wielomianu $p(z)$ leżą w obszarze

$$|z| \leq 1 + \frac{2\sin a}{|a_n|} \sum_{k=1}^{n} |a_k a_{k-1}|^{1/2}$$

РЕЗЮМЕ

Эта работа содержит некоторые результаты о расположению нулей полиномов. Первый относится к ограничению модуля нулей полинома $p(z) = \sum_{k=1}^{n+1} a_k z^{n-k+1}$, когда его коэффициенты исполняют неравенства $|a_{j+1}| \leq (1+a/j)|a_j|$. Потом доказано, что если коэффициенты полинома $p(z) = \sum_{k=0}^{n} a_k z^k$ исполняют условия $|\arg a_k - \beta| \leq a \leq \pi/2$

$$|a_0| \leq |a_1| \leq \ldots \leq |a_n|,$$

то все нули полинома $p(z)$ лежат в области

$$|z| \leq 1 + \frac{2\sin a}{|a_n|} \sum_{k=1}^{n} |a_k a_{k-1}|^{1/2}$$