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Convex Hulls of Some Classes of Univalent Functions

Otoczki wypukłe pewnych klas funkcji jednolistnych

Выпуклые оболочки некоторых классов однолистных функций

1. Introduction. Some forty years ago P. Montel [4] proposed studying properties of functions f, analytic and univalent in the unit disc D subject to the conditions

$$f(0) = 0,\ f^{(k)}(a) = 1,\ k = 0, 1, \ldots \tag{1.1}$$

where a, $0 < |a| < 1$, is a fixed point.

Functions satisfying these conditions with $k = 0$ have been investigated recently by many authors while there are a few results concerning others cases.

Recently L. Brickman, T. H. MacGregor and D. R. Wilken [1] and others developed a very interesting theory of so-called extreme points of a given family of analytic functions and gave many applications to extremal problems.

We will be concerned with classes of analytic functions that map D onto convex, starshaped or close-to-convex domains. We want here to establish some results concerning extreme points and convex hulls of classes of functions subject to (1.1) with either $k = 0$ or $k = 1$.

2. Main Results. We shall start with starlike functions. Let $\mathfrak{M}_a(a)$ denote the class of functions f analytic in D satisfying the conditions

$$f(0) = 0,\ f(a) = a,\ \operatorname{Re}\frac{zf'(z)}{f(z)} > a \tag{1.2}$$

where $0 \leqslant a < 1$, and let

$$\mathfrak{M}_a(0) = S_a^* = \left\{f\colon f(z) = z + a_2z^2 + \ldots,\ \operatorname{Re}\frac{zf'(z)}{f(z)} > a,\ |z| < 1\right\}.$$

We prove a formula which defines a one-to-one transformation of S_a^* onto $\mathfrak{M}_a(a)$.

Theorem 1. *If* $f \in S_\alpha^*$, *then*

$$F(z) = \frac{z(1-|a|^2)^{2(1-\alpha)}}{(z-a)(1-\bar{a}z)^{1-2\alpha}}\, f\left(\frac{z-a}{1-\bar{a}z}\right)$$

is in $\mathfrak{M}_\alpha(a)$, *and conversely.*

Proof. It is easy to see that $F(0) = 0$, $F(a) = a$. There is no loss of generality in assuming that f is analytic in closed unit disc. Then it is sufficient to check the condition $\operatorname{Re}\dfrac{zF'(z)}{F(z)} > \alpha$ on the unit circumference.

For, setting $\omega(z) = (z-a)(1-\bar{a}z)^{-1}$, $|z| = 1$ we have

$$\frac{zF'(z)}{F(z)} = \frac{-a(1-\bar{a}z)+z\bar{a}(z-a)(1-2\alpha)}{(z-a)(1-\bar{a}z)} + z\omega(z)\frac{F'(\omega(z))}{F(\omega(z))}$$

After some straightforward computations it gives

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > \frac{2\alpha\left(|a|^2-\operatorname{Re}(\bar{a}z)\right)}{|z-a|^2} + \alpha\frac{1-|a|^2}{|z-a|^2} = \alpha$$

Hence $G \in \mathfrak{M}_\alpha(a)$.

One can repeat the above considerations starting with a function F, $F \in \mathfrak{M}_\alpha(a)$ to end up with the conclusion that f,

$$f(z) = \frac{z}{(z+a)(1+\bar{a}z)^{1-2\alpha}}\, F\left(\frac{z+a}{1+\bar{a}z}\right), \quad |z|<1$$

is in S_α^*.

Theorem 1 has been proved.

Corollary 1. *The variability region of* $F(z)$ *for a fixed* z *and* F *ranging over the whole class* $\mathfrak{M}_\alpha(a)$ *is given by the inequality*

$$\left|\left(\frac{z}{F(z)}\right)^{1/2(1-\alpha)} - \frac{1-\bar{a}z}{1-|a|^2}\right| \leqslant \left|\frac{z-a}{1-\bar{a}z}\right|$$

Proof. It is easy to see that $\varphi(z) = z\left(\dfrac{f(z)}{z}\right)^{-\alpha+1}$ is in $\mathfrak{M}(0)$ iff f is in $\mathfrak{M}_0(0)$. The rest follows from Theorem 1 and the inequality $\left|\sqrt{\dfrac{z}{f(z)}}-1\right| \leqslant |z|$ which is due to A. Marx [3].

Corollary 2. *If $f(z) = z+\dots$ is a convex function in the unit disc D, then for each fixed point w in D the function*

$$g(z) = z\left(\frac{f(z)-f(w)}{z-w}\right)^{a}, \quad 0 \leqslant a \leqslant 2$$

is univalent and starlike of order $(1-a/2)$ in D.

Proof. If f satisfies the hypothesis then so does

$$\varphi(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right)-f(a)}{(1-|a|^2)f'(a)}$$

and, moreover $\operatorname{Re}\{zf'(z)/f(z)\} > \frac{1}{2}$.
Let us now apply Theorem 1 with $a = \frac{1}{2}$ to $\varphi(z)$. We have

$$F(z) = \frac{z}{z-a}\cdot\frac{f(z)-f(a)}{f'(a)} \in \mathfrak{M}_{\frac{1}{2}}(a)$$

Some simple computations yield

$$\frac{1}{2} < \operatorname{Re}\frac{zF'(z)}{F(z)} = \operatorname{Re}\left\{1-\frac{z}{z-a}+\frac{zf'(z)}{f(z)-f(a)}\right\}$$

which is equivalent to

$$a\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(a)}-\frac{z}{z-a}\right\} > -\frac{a}{2}$$

and the result follows.
The above corollary was known to hold for $a = 1, 2$ [7].

Corollary 3. *A necessary and sufficient condition for $f(z) = z+\dots$ to be convex in D is the inequality*

$$\operatorname{Re}\left\{\frac{2z_1 f(z_1)}{f(z_1)-f(z_2)}-\frac{z_1+z_2}{z_1-z_2}\right\} \geqslant 0$$

for any two points z_1, z_2 in D. [5].

Let $\mathscr{E}(\mathfrak{M}_a(a))$ denote the convex hull of the class $\mathfrak{M}_a(a)$, X stand for the unit circumference and let $\mathscr{P}$ be the set of all probability measures on X.

Theorem 2. *Suppose*

$$\mathscr{F} = \left\{f\colon f(z) = \int\limits_X \frac{z(1-ax)^{2(1-a)}}{(1-xz)^{2(1-a)}}\,d\mu(x), \mu\in\mathscr{P}, z\in D\right\}$$

then $\mathscr{F} = \mathscr{E}(\mathfrak{M}_a(a))$ *and the functions*

$$z \mapsto z(1-ax)^{2(1-a)}(1-xz)^{-2(1-a)} = k(z;x;a)$$

are the extreme points of $\mathfrak{M}_a(a)$.

Proof. The set $\mathscr{P}$ is convex and the mapping $\mathscr{P} \to \mathscr{F}$ defined by $\int_X k(z;x;a)\,d\mu(x)$ is linear. Hence the set $\mathscr{F}$ is convex, $\mathscr{F} \subset \mathscr{E}(\mathfrak{M}_a(a))$
Suppose $F \in \mathfrak{M}_a(a)$. Then there exists a function g in S_a^* such that (Theorem 1)

$$F(z) = \frac{z(1-|a|^2)^{2(1-a)}}{(z-a)(1-\bar{a}z)^{1-2a}}\, g\left(\frac{z-a}{1-\bar{a}z}\right)$$

This formula defines a linear mapping of S_a^* onto $\mathfrak{M}_a(a)$. The convex hull of S is given by the formula [2]:

$$g(\zeta) = \int_X \frac{\zeta}{(1-x\zeta)^{2-2a}}\,d\mu(x),\ \zeta \in D$$

Thus the functions

$$F(z) = (1-|a|^2)^{2(1-a)}\int_X \frac{z}{[1-\bar{a}z-x(z-a)]^{2(1-a)}}\,d\mu(x)$$

belong to $\mathscr{E}(\mathfrak{M}_a(a))$.
The formula $x = (y-a)(1-\bar{a}y)^{-1}$ defines a one-to-one mapping of X onto itself.
We put

$$d\mu(x) = d\mu\left(\frac{y-a}{1-y\bar{a}}\right) = d\nu(y)$$

where $\nu \in \mathscr{P}$, and we ultimately obtain

$$F(z) = \int_X \frac{z(1-ya)^{2(1-a)}}{(1-az)^{2(1-a)}}\,d\nu(y).$$

The uniquess of the extreme points follows from the fact that the transformation $\mathscr{P} \ni \mu \to F_\mu \in \mathfrak{M}_a(a)$ is one-to-one.
Theorem 2 has been proved.

Corollary 4. *The convex hull of the class of convex functions in* $\mathfrak{M}_0(a)$ *is given by the formula*

$$F(z) = \int_X \frac{z(1-xa)}{1-zx}\,d\mu(x).$$

Proof. It follows from the fact that the class of convex functions normalized by the conditions $F(0) = 0$, $F(a) = a$ is a subclass of $\mathfrak{M}_1(a)$ and, that the functions $z \to z(1-zx)^{-1}(1-ax)$, $|x| = 1$, are convex.

The above theorem may be used to obtain upper bounds for some functionals defined on $\mathfrak{M}_a(a)$.
This class is compact in a locally-convex topological space of all functions analytic in D so, according to a well-known Krein-Milman Theorem [6], any real linear functional on this class attains its supremum at an extreme point.
Hence we have

Corollary 5. *Suppose* $f(z) = \sum_1^\infty a_n z^n \in \mathfrak{M}_a(a)$. *Then there are the following sharp estimates*

(i) $|f(z)| \leqslant |z| \dfrac{1-|a|^2}{(|1-z\bar{a}|-|z-a|)^{2(1-a)}}$

(ii) $|f'(z)| \leqslant \max\limits_{|x|=1} k'(z;x;a)$

$\leqslant \dfrac{(1-|a|^2)^{2-2a}(1+|a|)[1+(1-2a)|z|]}{[|1-\bar{a}z|-|z-a|]^{3-2a}}$

(iii) $|a_1| \leqslant (1-|a|^2)^{2(1-a)}$

$$|a_n| \leqslant \frac{\prod_{k=2}^{n}(k-2a)}{(n-1)!}(1-|a|^2)^{2(1-a)}.$$

A function $F(z) = a_1 z + \ldots$ is said to be close-to-convex in the unit disc D if there exists a univalent convex function $\varphi(z) = b_1 z + \ldots$, $z \in D$ such that f'/φ' is a function of positive real part.

Denote by $\mathscr{M}_L(a)$ the whole class of such functions F normalized by the condition

$$F'(a) = 1, \ 0 < |a| < 1.$$

Theorem 3. *The set* $\mathscr{E}(\mathscr{M}_L(a))$ *consists of functions of the form*

$$F(z) = \int_{X \times X} \frac{z - \frac{x+y}{2} z^2}{(1-yz)^2} \cdot \frac{(1-ay)^3}{1-ax} \, d\mu(x, y)$$

where $\mu(x, y)$ *is a probability measure on* $X \times X$.
The extreme points of $\mathscr{M}_L(a)$ *are precisely functions*

$$D \ni z \mapsto \frac{z - \frac{x+y}{2} z^2}{(1-yz)^2} \cdot \frac{(1-ay)^3}{1-ax} \in \mathscr{M}_L(a)$$

Proof. Let us notice first that if $f(z) = z+\dots$ is a close-to-convex function, then F defined by

$$(*) \qquad F(z) = (1-|a|^2)\left[f\left(\frac{z-a}{1-\bar{a}z}\right) - f(-a)\right], \quad F(a) = 1$$

is in $\mathcal{M}_L(a)$.
To see this is sufficient to notice that if f'/φ' has positive real part in D so does F'/h', where $h = \varphi\left(\frac{z-a}{1-\bar{a}z}\right)$, φ being convex. It has been proved [1] that close-to-convex function normalized by the conditions $f(0) = f'(0)-1 = 0$ have the representation

$$(**) \qquad f(z) = \int\limits_{X\times X} \frac{z - \frac{x+y}{2}z^2}{(1-yz)^3}\, d\mu(x, y).$$

Suppose $F \in \mathcal{M}_L(a)$. Thus by $(*)$ there exists exactly one close-to-convex function $f(z) = z+\dots$ such that

$$F(z) = (1-|a|^2)\left[f\left(\frac{z-a}{1-\bar{a}z}\right) - f(-a)\right]$$

By $(**)$ there exists exactly one probability measure on $X\times X$ for which derivatives of functions F in the convex hull of $\mathcal{M}_L(a)$ are of the form

$$(***) \qquad F'(z) = (1-|a|^2)\int\limits_{X\times X} \frac{1-\bar{a}z - x(z-a)}{1-\bar{a}z - y(z-a)^3}\, d\mu(x, y)$$

It follows by using the transformations

$$x \mapsto (x+\bar{a})(1+ax)^{-1}, \quad y \mapsto (y+\bar{a})(1+ay)^{-1}$$

which map X^2 onto itself that $(***)$ is equivalent to

$$F(z) = \int\limits_{X\times X} \frac{(1-xz)(1-ay)^3}{(1-yz)^3(1-ax)}\, d\mu(x, y)$$

The rest follows by integrating and making use of the Fubini's theorem. Each function of the integrand in Theorem 3 belongs to $\mathcal{M}_L(a)$ and the set of probability measures is convex. Theorem 3 has been proved.

Let us notice, that the above considerations are valid in the case when $F(z) = a_1 z+\dots$, $F'(a) = 1$ belongs to the class of convex and univalent functions in D.
It gives us

Theorem 4. *Suppose* $\mathcal{M}_K(a)$ *is the class of convex functions* $F(z) = a_1 + \ldots$, $F'(z) = 1$, $z \in D$. *Then* $\mathcal{E}(\mathcal{M}_K(a))$ *consists of functions of the form*

$$F(z) = \int_X \frac{z(1-ax)^2}{1-xz} d\mu(x),$$

where μ *is a probability measure on* X.
The extreme points of the class $\mathcal{M}_K(a)$ *are precisely the functions*

$$z \mapsto z(1-ax)^2(1-xz)^{-1}, \quad |x| = 1.$$

The last two theorems yield sharp upper bounds for $|F(z)|$, $|F(z)|$ and $|a_n|$.

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STRESZCZENIE

W pracy tej wyznaczono otoczki wypukłe i punkty ekstremalne dla klas $\mathfrak{M}_a^*(a)$ i $\mathcal{M}_L(a)$ funkcji jednolistnych w kole jednostkowym D, gdzie: $\mathfrak{M}_a^*(a)$ oznacza klasę funkcji f a-gwiaździstych z unormowaniem $f(0) = 0$, $f(a) = a$, $\mathcal{M}_L(a)$ oznacza klasę funkcji F prawie wypukłych z unormowaniem $F(0) = 0$, $F'(a) = 1$ ($|a| < 1$, $0 \leqslant a < 1$)

РЕЗЮМЕ

В этой работе получено выпуклые оболочки и экспериментальные точки в классах $\mathfrak{M}_a^*(a)$ и $\mathcal{M}_L(a)$ однолистных функций в D, где $\mathfrak{M}_L^*(a)$ обозначает класс функции f,a-звездных для которых $f(0) = 0$, $f(a) = a$,$\mathcal{M}_L(a)$ — класс F почти выпуклых функций с нормировкой $F(0) = 0$, $F'(a) = 1$.