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On the Full Solution of the Functional-Paratingent Equation

O pełnym rozwiązaniu równania paratyngensowo-funkcjonalowego

О полном решении функционально-паратингентного уравнения

This note concerns the existence of a full solution of the functional-paratingent equation. We present the generalisation of our earlier notes [3] and [4] in which we considered the problem of existence of a solution for a paratingent equation with deviated argument.

I. Notations and definitions

Let us accept the following symbols.

$p < 0$ is a fixed number belonging to the real line R. $R^+ = [0, \infty) \subset R$. R^m denotes a m-dimensional Euclidean space with the norm $|x| = \max = \max(|x_1|, \ldots, |x_m|)$, where $x = (x_1, \ldots, x_m)$.

$\operatorname{Conv} R^m$ is the family of all convex compact and nonempty subsets of R^m with the distance between them being understood in the Hausdorff sense.

C is the space of all continuous functions $g\colon [p, \infty] \to R^m$ with topology defined by an almost uniform convergence. It is well-known that C is a metrizable locally convex linear topological space. $[\varphi]_t$, $t \geqslant p$, denotes the function φ which is localized within the interval $[p, t]$ and $\|\varphi\|_t = \max\limits_{p \leqslant s \leqslant t} \varphi(s)$ (i.e. $[\varphi]_t$ is the best non-decreasing majorant of φ on $[p, t]$).

$\mathfrak{C}$ is the space of all functions $[\varphi]_t$, where $\varphi \in C$ and $t \geqslant p$, with the metric being understood as a distance of graph (the graph being a subset of $R \times R^m$) of these functions in the Hausdorff sense (a so-called graph topology).

Having a function $\varphi \in C$ and $t \geqslant p$ the set of all limit points

$$x = \frac{\varphi(t_i) - \varphi(s_i)}{t_i - s_i}$$

where s_i, $t_i \geqslant p$, $s_i \to t$, $t_i \to t$ and $s_i \neq t_i$, $i = 1, 2, \ldots$ will be called paratingent of φ at the point t and denoted by $(P\varphi)(t)$.

Taking only the limit points for which $t \leqslant s_i$, $t \leqslant t_i$ and $t_i \to t$, $s_i \to t$, $t_i \neq s$ one obtains the right-hand paratingent $(P^+\varphi)(t)$ of φ at the point t.

Let $F\colon R^+ \times \mathfrak{C} \to \operatorname{Conv} R^m$ be a continuous mapping, let $\nu\colon R^+ \mapsto R^+$ be a continuous function such that $\nu(t) > t$ and let $[\xi]_0 \in \mathfrak{C}$. We shall deal with the functional-paratingent equation

$$(1)\qquad (Px)(t) \subset F(t, [x]_{\nu(t)}), \; t \geqslant 0$$

with the initial condition

$$(2)\qquad x(t) = \xi(t), \; p \leqslant t \leqslant 0.$$

By the full solution of (1), which satisfies the condition (2), we mean any function $\varphi \in C$ such that

$$(P\varphi)(t) = F(t, [\varphi]_{\nu(t)}), \; t > 0$$

$$(P^+\varphi)(0) = F(0, [\varphi]_{\nu(t)})$$

and

$$\varphi(t) = \xi(t), \quad p \leqslant t \leqslant 0.$$

Put $\|F(t, [x]_v)\| = \sup\{|z|\colon z \in F(t, [x]_v), \, (t, [x]_v) \in R^+ \times \mathfrak{C}\}$ and let a and A be fixed constans such that $0 < a \leqslant 1$ and $A \geqslant \max[1, [\xi]_0]$.

II. Theorem. *If the mapping F satisfies the condition*

$$(3)\qquad \|F(t, [x]_v)\| \leqslant M(t) + N(t)([x]_v)^a,$$

where the functions M and N are non-negative and continuous, and if the function ν satisfies the inequality

$$(4)\qquad a\Lambda(\nu(t)) \leqslant \Lambda(t) + e^{-1} \quad \text{for } t \in R^+$$

where

$$\Lambda(t) = \int_0^t [M(s) + N(s)]ds,$$

then there exists a full solution of equation (1) *which satisfies the initial condition* (2)*. Moreover this solution satisfies the inequality*

$$(5)\qquad |\varphi(t)| \leqslant A\exp[e\Lambda(t)] \quad \text{for } t \in R^+.$$

First we shall prove some Lemmas.

III. Lemma 1. *Suppose that the function $f\colon R^+ \times \mathfrak{C} \mapsto R^m$ satisfies the following conditions*

(i) *for each fixed $t \in R^+$, f is continuous in respect to $[x]_v$, $[x]_v \in \mathfrak{C}$,*

(ii) *for each fixed $[x]_v \in \mathfrak{C}$, f is Lebesque measurable with respect to $t \in R^+$,*

(iii) $|f(t, [x]_v| \leqslant M(t) + N(t)([x]_v)^a$ *for each* $(t, [x]_v) \in R^+ \times \mathfrak{C}$.

Then there exists at least one solution in the Caratheodory sense of the equation

$$x'(t) = f(t, [x]_{\nu(t)}), \quad t \geqslant 0 \tag{6}$$

which satisfies the initial condition (2) *for* $p \leqslant t \leqslant 0$ *and, moreover, the inequality* (5) *for* $t \geqslant 0$.
(*By solution in the Caratheodory sense of* (6) *we mean any absolutely continuous function* φ: $R^+ \mapsto R^m$ *satisfying* (6) *almost everywhere in* R^+).

Proof. Let K denote a family of all functions belonging to C and satisfying the following three conditions

$$|\varphi(t)| \leqslant A \exp[eA(t)] \quad \text{for } t \geqslant 0, \tag{7}$$

$$|\varphi(t+h) - \varphi(t)| \leqslant A \int_t^{t+h} \{\exp[eA(s)]\} ds \quad \text{for } t \geqslant 0, \text{ and } h > 0, \tag{8}$$

$$\varphi(t) = \xi(t) \quad \text{for } p \leqslant t \leqslant 0. \tag{9}$$

We see at once that this family is a nonempty compact and convex subset of the space C.

Let us consider the operator D: $C \mapsto C$ defined by formula

$$(D\varphi)(t) = \begin{cases} \xi(t) & \text{for } p \leqslant t \leqslant 0, \\ \xi(0) + \int_t^0 f(s, [\varphi]_{\nu(s)}) ds & \text{for } t \geqslant 0. \end{cases}$$

At first we shall show that D is continuous. Let $\varphi, \varphi_i \in C$ and $\varphi_i \to \varphi$, $i = 1, 2, \ldots$. Let us fix $T \geqslant 0$. Then the sequence $\{\varphi_i\}$ is uniformly convergent to a function φ on the interval $[0, T^*]$, where $T^* = \max_{0 \leqslant t \leqslant T} \nu(t)$.

Let us denote

$$B = \sup_i (\|\varphi\|_{T^*})^a$$

$$w_i(t) = f(t, [\varphi_i]_{\nu(t)}), \quad 0 \leqslant t \leqslant T \qquad i = 1, 2, \ldots$$
$$v(t) = M(t) + BN(t), \quad 0 \leqslant t \leqslant T$$

Each of the functions w_i is integrable on $[0, T]$.
Furthermore, for each $t \in [0, T]$ we have

$$|w_i(t)| \leqslant v(t)$$

and

$$w_i(t) \to w(t) = f(t, [\varphi]_{\nu(t)}), \quad i = 1, 2, \ldots$$

Therefore, in view of well known theorems in the theory of real functions,

$$\int_0^T |w_i(t)-w(t)|\,dt \to 0, \quad i = 1, 2, \ldots.$$

On the other hand for each $t \in [0, T]$

$$|(D\varphi_i)(t)-(D\varphi)(t)| \leqslant \int_0^t |w_i(s)-w(s)|\,ds$$

$$\leqslant \int_0^T |w_i(t)-w(t)|\,dt \to 0, \quad i = 1, 2, \ldots$$

Hence it follows that the sequence $\{D\varphi_i\}$ uniformly converges to a function $D\varphi$ on $[0, T]$. Since T was arbitrary and $(D\varphi_i)(t) = \xi(t)$ for $p \leqslant t \leqslant 0$, the sequence of functions $\{D\varphi_i\}$ is uniformly convergent to $D\varphi$ on each compact subinterval of interval $[p, \infty)$. Thus $D\varphi_i \to D\varphi$ in the space C. This means that the operator D is continuous. Besides D maps the set $K \subset C$ into itself.

Indeed, if $\varphi \in K$, then firsty $(D\varphi)(t) = \xi(t)$ for $p \leqslant t \leqslant 0$, and by conditions (3), (8) and (4) we have

$$|(D\varphi)(t+h)-(D\varphi)(t)| \leqslant \int_t^{t+h} |f(s, [\varphi]_{\nu(s)})|\,ds \leqslant$$

$$\leqslant \int_t^{t+h} [M(s)+N(s)([\varphi]_{\nu(s)})^a]\,ds \leqslant \int_t^{t+h} \left\{M(s)+A^a N(s)\left(\exp\left[e\Lambda\left(\nu(s)\right)\right]\right)^a\right\} ds \leqslant$$

$$\leqslant A \int_t^{t+h} \left\{L(s)\exp\left[ae\Lambda\left(\nu(s)\right)\right]\right\} ds \leqslant \int_t^{t+h} \{L(s)\exp[e\Lambda(s)+1]\,ds$$

$$= A \int_t^{t+h} \left\{\exp\left[e\Lambda(s)\right]\right\}' ds \quad \text{for } t \geqslant 0 \text{ and } h > 0,$$

where $L(s) = M(s)+N(s)$.
Hence we obtain

$$|(D\varphi)(t)| \leqslant |(D\varphi)(0)| + A \int_0^t \{\exp[e\Lambda(s)]\}' ds \leqslant$$

$$\leqslant A + A\{\exp[e\Lambda(t)]-1\} = A\exp[e\Lambda(t)] \quad \text{for } t \geqslant 0.$$

Consequently $D\varphi \in K$.

So we see that the operator D fulfills all the hypotheses of the well-known Schauder's-Tichonov's theorem on a fixed point. Therefore, there exists a function $\varphi \in K$ such that $\varphi = D\varphi$ what means that

$$\varphi'(t) = f(t, [\varphi]_{v(t)}) \quad \text{for almost every } t \geqslant 0$$

$$\varphi(t) = \xi(t) \quad \text{for } p \leqslant t \leqslant 0$$

and obviously

$$|\varphi(t)| \leqslant A \exp[e\Lambda(t)] \quad \text{for } t \geqslant 0.$$

Our lemma is thus proved.

Lemma 2. *There exists a sequence of sets* $A \subset R$, $n = 0, 1, 2, \ldots$ *such that*

(10) $$A \cap A = \emptyset \quad \text{if} \quad i \neq j,$$

(11) $$\bigcup_{n=0}^{\infty} A_n = R^+$$

(12) $$\bigwedge_{(a,b)\subset R^+} \mu\big((a, b) \cap A_n\big) > 0 \quad \textit{for} \quad n = 0, 1, 2, \ldots$$

μ being the Lebesque measure.

Proof. By lemma 1 in [3] there exists a sequence of sets $B_n \subset (0,1)$ such that

a) $$B_i \cap B_j = \emptyset \text{ if } i \neq j,$$

b) $$\bigcup_{n=0}^{\infty} B_n = [0, 1)$$

c) $$\bigwedge_{(\alpha,\beta)\subset[0,1)} \mu[(\alpha, \beta) \cap B_n] > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Now let us put

$$A_n = \eta(B_n), \; n = 0, 1, \ldots,$$

where $\eta\colon [0, 1) \to R^+$ is a function defined by $\eta(t) = \operatorname{tg}\frac{\pi}{2} t$ for $t \in [0, 1)$.

Taking advantage of the properties a) – c) it can easily be shown that the sets A_n satisfy (10) – (12).
Indeed:

$$A_i \cap A_j = \eta(B_i) \cap \eta(B_j) = \eta(B_i \cap B_j) = \eta(\emptyset) = \emptyset \text{ if } i \neq j,$$

$$\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \eta(B_n) = \eta\Big(\bigcup_{n=0}^{\infty} B_n\Big) = \eta\big([0, 1)\big) = R^+.$$

To prove (12) let us notice that for arbitrary interval $(a, b) \subset R^+$ there exists exactly one interval $(\alpha, \beta) \subset [0, 1)$ such that $\eta\big((\alpha, \beta)\big) = (a, b)$. Then

$$\mu\big((a, b) \cap A_n\big) = \mu\big(\eta\big((\alpha, \beta)\big) \cap \eta(B_n)\big) = \mu\big(\eta[(\alpha, \beta) \cap B_n]\big)$$
$$= \int_{(\alpha, \beta) \cap Bn} \eta'(s)\, ds > 0, \quad n = 0, 1, 2, \ldots,$$

as $\mu[(\alpha, \beta) \cap B_n] > 0$, which completes the proof of the lemma.

Lemma 3. If the absolutely continuous function $g\colon R^+ \mapsto R^m$ satisfies the condition

(13) $$g'(t) \subset F(t, [g]_{\nu(t)}) \text{ for almost every } t \geqslant 0$$

then

(14) $$(Pg)(t) \subset F(t, [g]_{\nu(t)}) \text{ for every } t > 0$$

and

(15) $$(P^+ g)(0) \subset F(0, [g]_{\nu(0)}).$$

The proof is omitted because it is analogical to the proof of the Lemma 2 in [3].

IV. The proof of the theorem. Let A_n, $n = 0, 1, 2, \ldots$, be a sequence of sets satisfying (10) — (12). By lemma 5.2 in [1] there exists a sequence of continuous selections $f_n\colon R^+ \mapsto R^m$, $n = 0, 1, 2, \ldots$, such that $f_n(t, [x]_v) \in F(t, [x]_v)$ for every $(t, [x]_v) \in R^+ \times \mathfrak{C}$, $n = 0, 1, 2, \ldots$, and the set $\{f_n(t, [x]_v)\}_{n=0}^{\infty}$ is dense in $F(t, [x]_v)$ for each $(t, [x]_v) \in R^+ \times \mathfrak{C}$.
Let us put

$$f(t, [x]_v) = f_n(t, [x]_v) \quad \text{if} \quad (t, [x]_v) \in A_n \times \mathfrak{C}.$$

The function f has the following properties
a) for each fixed $t \in R^+$, f is continuous in $[x]_v$, $[x]_v \in \mathfrak{C}$,
b) for each fixed $[x]_v \in \mathfrak{C}$, f is Lebesque measurable with respect to $t \in R^+$,
c) $|f(t, [x]_v)| \leqslant M(t) + N(t)([x]_v)$ for every $(t, [x]_v) \in R^+ \times \mathfrak{C}$.
The properties a) and c) do not need to be explained. To shown b) let us notice that f can be written in the following form

$$f(t, [x]_v) = \sup_n g_n(t, [x]_v) + \inf_n g_n(t, [x]_v),$$

where

$$g_n(t, [x]_v) = \begin{cases} f_n(t, [x]_v), & \text{if } t \in A_n, \\ 0, & \text{if } t \notin A_n, \end{cases}$$

and $\sup_n g_n(\,,\,) = \big(\sup_n g_n^1(\,,\,), \ldots, \sup_n g_n^m(\,,\,)\big)$ (analogically $\inf_n$).

Now if is easy to see that f is measurable because all the functions g_n are measurable.

Therefore, by lemma 1, there exists a function $\varphi \in C$ such that

$$\varphi'(t) = f(t, [\varphi]_{\nu(t)}) \quad \text{a.e. in } R^+$$

and

$$\varphi(t) = \xi(t) \quad \text{for} \quad p \leqslant t \leqslant 0.$$

Hence

$$\varphi'(t) \in F(t, [\varphi]_{\nu(t)}) \quad \text{a.e. in } R^+$$

and, as before,

$$\varphi(t) = \xi(t) \quad \text{for } p \leqslant t \leqslant 0.$$

By lemma 3 we have

$$(P\varphi)(t) \subset F(t, [\varphi]_{\nu(t)}) \quad \text{for every } t > 0$$

and

$$(P^+\varphi)(0) \subset F(0, [\varphi]_{\nu(0)}).$$

Now we shall prove that $F(t, [\varphi]_{\nu(t)}) \subset (P\varphi)(t)$, $t > 0$,

$$F(0, [\varphi]_{\nu(0)}) \subset (P^+\varphi)(0).$$

To do this let us fix $\bar{t} \geqslant 0$ and choose arbitrary $z \in F(\bar{t}, [\varphi]_{\nu(\bar{t})})$. Since the set $\{f_n(\bar{t}, [\varphi]_{\nu(\bar{t})})\}_{n=0}^{\infty}$ is dense in $F(\bar{t}, [\varphi]_{\nu(\bar{t})})$ we can choose a subsequence f_{n_k}, $k = 0, 1, 2, \ldots$, such that

$$|f_{n_k}(\bar{t}, [\varphi]_{\nu(\bar{t})}) - z| < 2^{-k}. \tag{16}$$

On the other hand, from the continuity of the functions f_n and measurable density of the sets A_n (cf. (12)) it follows that there exists a sequence $t_k \in R^+$, $k = 0, 1, 2, \ldots$, satisfying the following conditions

$$t_k \in A_k, \ \lim_{k\to\infty} t_k = \bar{t}$$

$$\varphi'(t_k) = f_{n_k}(t_k, [\varphi]_{\nu(t_k)}) \tag{17}$$

and

$$|f_{n_k}(t_k, [\varphi]_{\nu(t_k)}) - f_{n_k}(\bar{t}, [\varphi]_{\nu(\bar{t})})| < 2^{-k}, \quad k = 0, 1, 2, \ldots. \tag{18}$$

Now in view of (17) we can choose another sequence $s_k \in R^+$, $k = 0, 1, 2, \ldots$, such that $|s_k - t_k| < 2^{-k}$, $s_k \neq t_k$ and

$$\left| \frac{|\varphi(s_k) - \varphi(t_k)}{s_k - t_k} - f_{n_k}(t_k, [\varphi]_{\nu(t_k)}) \right| < 2 \ , \quad k = 0, 1, 2, \ldots$$

From (16) and (18) we shall obtain

$$\left|\frac{\varphi(s_k)-\varphi(t_k)}{s_k-t_k}-z\right|<3\cdot 2^{-k}, \quad k=0,1,2,\ldots$$

which means that $z\in(P\varphi)(\bar{t})$ (or $z\in(P^{+}\varphi)(0)$ if $\bar{t}=0$).
Thus $F(\bar{t},[\varphi]_{\nu(\bar{t})})\subset(P\varphi)(\bar{t})$ (or $F(0,[\varphi]_{\nu(0)})\subset(P^{+}\varphi)(0)$).
Since $\bar{t}$ was arbitrary, we have

$$F(t,[\varphi]_{\nu(t)})\subset(P\varphi)(t),\ t>0$$

$$F(0,[\varphi]_{\nu(0)})\subset(P\ \ \varphi)(0).$$

Finally there is

$$(P\varphi)(t)=F(t,[\varphi]_{\nu(t)}) \quad \text{for } t>0,$$

$$(P^{+}\varphi)(0)=F(0,[\varphi]_{\nu(0)})$$

and obviously

$$\varphi(t)=\xi(t) \quad \text{for} \quad p\leqslant t\leqslant 0,$$

which completes the proof of our theorem.

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STRESZCZENIE

W pracy udowodniono twierdzenie o istnieniu pełnego rozwiązania równania paratyngensowego z przesuniętym argumentem

(1) $$(Px)(t)\subset F(t,[x]_{\nu(t)}),\ 0\leqslant t,$$

z warunkiem początkowym

(2) $$x(t)=\xi(t),\ p\leqslant t\leqslant 0.$$

РЕЗЮМЕ

В работе доказана теорема о существовании полного решения паратингентного уравнения с отклоняющим аргументом

(1) $$(Px)(t) \subset F(t, [x]_{\nu(t)}), \; 0 \leqslant t,$$

с начальным условием

(2) $$x(t) = \xi(t), \; p \leqslant t \leqslant 0.$$

Полным решением уравнения (1), удовлетворяющим условию (2) называем каждую функцию $\varphi \in C$ такую, что $(P\varphi)(t) = F(t, [\varphi]_{\nu(t)})$, $0 \leqslant t$.