ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXIX, 7 SECTIO A 1975

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

MARIA FAIT, JAN STANKIEWICZ, JADWIGA ZYGMUNT

# On Some Classes of Polynomials

O pewnych klasach wielomianów

Об некоторых классах полиномов

Let $P_{n_a}$ denote a class of polynomials of the form:

$$p(z) = 1 + a_1 z + \ldots + a_n z^n$$

which satisfy the condition

$$\operatorname{Re} p(z) > a, \text{ for } z \in K_1 = \{z\colon |z| < 1\},$$

where $n$ is a fixed positive integer and $a$ is a fixed real number belonging to the interval $\langle 0, 1)$.

F. Holland [2] proved the following

**Theorem A.** *If* $p(z) = 1 + a_1 z + \ldots + a_n z^n \in P_{n,0}$ *then*

$$(1) \qquad |a_k| \leqslant 2\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad k = 1, 2, \ldots, n,$$

where $\left[\frac{n}{k}\right]$ is the greatest integer $\leqslant \frac{n}{k}$. The estimate (1) is sharp.

Long before (1928) E. Egervary and O. Szasz [1] proved

**Theorem B.** *Let* $Q(\theta) = 1 + \sum_{k=1}^{n} (a_k \cos k\theta + \beta_k \sin k\theta)$ *be a non-negative trigonometrical polynomial. Then*

$$\sqrt{a_k^2 + \beta_k^2} \leqslant 2\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad k = 1, 2, \ldots, n$$

*and the estimate is sharp.*

**Remark 1.** Theorem A is a simple corollary of Theorem B. It is enough to remark that if we put $z = e^{i\theta}$, $a_k = \alpha_k - i\beta_k$, $k = 1, 2, \ldots, n$, then

$$Q(\theta) = \operatorname{Re} p(z) = \operatorname{Re} p(e^{i\theta}).$$

Thus the condition $\operatorname{Re} p(z) > 0$, for $|z| < 1$, implies that $\operatorname{Re} p(z) \geqslant 0$ for $|z| = 1$ and by this $Q(\theta) \geqslant 0$. Furthermore, $|a_k| = |\bar{a}_k| = \sqrt{\alpha_k^2 + \beta_k^2}$.

The Theorems A and B may be generalized on the classes $P_{n,a}$.

**Theorem 1.** *If a polynomial*

$$w(z) = 1 + a_1 z + \ldots + a_n z^n \in P_{n,a}$$

*then*

$$(2) \qquad |a_k| \leqslant 2(1-a)\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad k = 1, 2, \ldots, n.$$

*The estimate is sharp. The extremal polynomials, which give equality in (2) have a form:*

$$(3) \qquad w_\xi(z) = q(z)\left\{1 + (1-a)\sum_{l=1}^{\left[\frac{n}{k}\right]} \tilde{a}_l (e^{i\varphi} z)^{kl}\right\}$$

*where*

$$\tilde{a}_l = \frac{2}{\left[\frac{n}{k}\right]+2}\left(\frac{n}{k} - l + 1\right)\cos\frac{l\pi}{\left[\frac{n}{k}\right]+2} + \sin\frac{(l+1)\pi}{\left[\frac{n}{k}\right]+2} \Big/ \sin\frac{\pi}{\left[\frac{n}{k}\right]+2}$$

*and $q(z)$ is a polynomial whose degree is $\leqslant n - [n/k]k$ $(n - [n/k]\cdot k \in \{0, 1, \ldots, k-1\})$ and chosen so that the polynomial $w_\xi(z)$ given by (3) satisfies the conditions of our theorem. For $n \to \infty$ we obtain the result of Libera [5].*

**Proof.** We first remark that $w(z) \in P_{n,a}$ if and only if $p(z) = (w(z) - a)/(1-a) \in P_{n,0}$.

If we put $w(z) = 1 + a_1 z + \ldots + a_n z^n$, $p(z) = 1 + b_1 z + \ldots + b_n z^n$ then the coefficients $a_k, b_k$ are related by the equality $a_k = (1-a) b_k$. The polynomial $p(z)$ satisfies the conditions of Theorem A and therefore (1) implies (2).

In the paper [1] p. 646, the extremal trigonometrical polynomials in Theorem B are determined. Using these extremal polynomials and the relation

$$(4) \qquad w(z) = a + (1-a)p(z)$$

between the classes $P_{n,a}$ and $P_{n,0}$ one can find that the extremal polynomials have the form (3).

Basing on the results of papers [1], [2] we can find the estimates of some functionals such as: $\max\limits_{|z|\leqslant 1} |w(z)|$, $|a_k|+|a_{n-k+1}|$ and $\max\limits_{|z|\leqslant 1} |\mathrm{Re}\{izw'(z)\}| = \max\limits_{|z|\leqslant 1} |\mathrm{Im}\{zw'(z)\}|$ for the class $P_{n,a}$.

**Theorem 2.** *Let* $w(z) = 1+a_1z+\ldots+a_nz^n \epsilon P_{n,a}$.
*Then*

$$|a_k|+|a_{n-k+1}| \leqslant 2(1-a), \quad k = 1, 2, \ldots, n, \tag{5}$$

$$\max_{|z|\leqslant 1} |w(z)| \leqslant 1+(1-a)n \tag{6}$$

$$\max_{|z|\leqslant 1} |\mathrm{Re}\{izw'(z)\}| = \max |\mathrm{Im}\{zw'(z)\}| \leqslant (1-a)\sqrt{\frac{n+1}{2}\binom{n+2}{3}}. \tag{7}$$

*The estimate* (6) *is sharp. The extremal polynomials have the form*

$$w_\xi(z) = 1+2(1-a)\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)(e^{i\varphi}z)^k \tag{8}$$

*where* $\varphi$ *is an arbitrary fixed real number.*

**Proof.** The inequality (5) follows from (4) and from inequality (22) of [1], p. 650. The inequality (7) follows from (4) and from inequality (23) of [1], p. 651. Next, the inequality (6) follows from (4) and from Theorem 2 of [2], p. 54.

With the class $P_{n,a}$ we can connect some classes of univalent polynomials.
Denote by $R_{n,a}$ the class of univalent polynomials

$$f(z) = z+a_2z^2+\ldots+a_nz^n$$

satisfying the condition

$$\mathrm{Re} f'(z) > a \quad \text{for } z \epsilon K_1.$$

Thus we have

$$f \epsilon R_{n,a} \Leftrightarrow f(0) = 0 \wedge f' \epsilon P_{n-1,a}. \tag{9}$$

Let $H_{n,a}$ be the class of univalent polynomials

$$h(z) = z+a_2z^2+\ldots+a_nz^n$$

such that $\mathrm{Re}[zh'(z)]' > a$ for $z \epsilon K_1$. In this case we have

$$h \epsilon H_{n,a} \Leftrightarrow h(0) = 0 \wedge [zh'(z)]' \epsilon P_{n-1,a}. \tag{10}$$

Furthermore, denote by $G_{n,a}$ the class of univalent polynomial $g(z) = z+a_2z^2+\ldots+a_nz^n$ such that $\mathrm{Re}\{g'(z)+\frac{1}{2}zg''(z)\} > a$ for $z\in K_1$. Now

(11) $$g\in G_{n,a} \Leftrightarrow g(0) = 0 \wedge \{g'(z)+\tfrac{1}{2}zg''(z)\}\in P_{n-1,a}$$

**Theorem 3.** *If $f(z) = z+a_2z^2+\ldots+a_nz^n\in R_{n,a}$ then*

(12) $$|a_k| \leqslant \frac{2(1-a)}{k}\cos\frac{\pi}{\left[\frac{n-1}{k-1}\right]+2}, \quad k = 2, 3, \ldots, n$$

*This estimate is sharp. The equality occurs only for the polynomials of the form*

(13) $$f_\xi(z) = \int_0^z w_\xi(\zeta)d\zeta,$$

*where $w_\xi(z)$ is given by* (3).

**Remark 2.** If $n$ approaches $\infty$ then we obtain the estimates of coefficients for the class $R_a$ of univalent functions $(\mathrm{Re}f'(z) > a)$ Namely, if $f(z) = z+\sum_{k=2}^{\infty}a_kz^k$ and $\mathrm{Re}f'(z) > a$, then $|a_k| \leqslant \frac{2(1-a)}{k}$.

For $a = 0$ and $n$ approaches $\infty$, we obtain the classical result, that the coefficients of the functions with bounded rotation are dominated by $2/k$ (see e.g. [6]).

**Proof of Theorem 3.** By the relation (9) between the classes $P_{n-1,a}$, $R_{n,a}$ we have $f\in R_{n,a}$ if and only if there exists a polynomial $w\in P_{n-1,a}$ such that

(14) $$f(z) = \int_0^z w(\zeta)d\zeta.$$

Therefore if $f(z) = z+a_2z^2+\ldots+a_nz_n$ and $w(z) = 1+\tilde{a}_1z+\ldots+\tilde{a}_{n-1}z^{n-1}$, then $a_k = \tilde{a}_{k-1}/k$, $k = 2, 3, \ldots, n$.

Using the estimate (2) for $\tilde{a}_{k-1}$ we have finally

$$|a_k| \leqslant \frac{2(1-a)}{k}\cos\frac{\pi}{\left[\frac{n-1}{k-1}\right]+2}.$$

The equality occurs in (12) only if the equality occurs in (2). This together with (9) yields (13).

**Theorem 4.** *If $h(z) = z+a_2z^2+\ldots+a_nz^n\in H_{n,a}$, then*

(15) $$|a_k| \leqslant \frac{2(1-a)}{k^2}\cos\frac{\pi}{\left[\frac{n-1}{k-1}\right]+2}, \quad k = 2, 3, \ldots, n.$$

*The equality occurs in* (15) *only for the polynomials of the form*

$$h_\xi(z) = \int_0^z \left\{ \frac{1}{\zeta} \int_0^\zeta w_\xi(\eta)\, d\eta \right\} d\zeta = \int_0^z \frac{f_\xi(\zeta)}{\zeta}\, d\zeta, \tag{16}$$

*where* $w_\xi(z)$ *is given by* (3) *and* $f_\xi(z)$ *is given by* (13).

**Proof.** The relation (10) between the classes $P_{n-1,a}$ and $H_{n,a}$ may be written in the form

$$h(z) = \int_0^z \left\{ \frac{1}{\zeta} \int_0^\zeta w(\eta)\, d\eta \right\} d\zeta$$

and by this, if $h(z) = z + a_2 z^2 + \ldots + a_n z^n$ and $w(z) = 1 + \tilde{a}_1 z + \ldots + \tilde{a}_{n-1} z^{n-1}$ then

$$a_k = \tilde{a}_{k-1}/k^2. \tag{17}$$

Now (17) and (2) implies (15). The equality occurs in (15) only if the equality occurs in (2) and therefore by (3) and (10) we obtain (16).

**Theorem 5.** *If* $g(z) = z + a_2 z^2 + \ldots + a_n z^n \in G_{n,a}$, *then*

$$|a_k| \leqslant 4\, \frac{1-a}{k(k+1)} \cos \frac{\pi}{\left[\frac{n-1}{k-1}\right] + 2}. \tag{18}$$

*The equality occurs in* (18) *only for the polynomials* $g_\xi(z)$ *of the form*

$$g_\xi(z) = \frac{2}{z} \int_0^z f_\xi(\zeta)\, d\zeta = \frac{2}{z} \int_0^z \left\{ \int_0^\zeta w_\xi(\eta)\, d\eta \right\} d\zeta, \tag{19}$$

*where* $w_\xi(z)$ *is given by* (3) *and* $f_\xi(z)$ *is given by* (13).

**Proof.** Similarly as above one can remark that if $g(z) = z + a_2 z^2 + \ldots + a_n z^n$ and $w(z) = 1 + \tilde{a}_1 z + \ldots + \tilde{a}_{n-1} z^{n-1}$ are connected by (11), then

$$a_k = \frac{2}{k(k+1)}\, \tilde{a}_{k-1}. \tag{20}$$

Hence, by (20), (11) and (2) we obtain (18) and the formula (19) for the extremal polynomials.

Now, we will consider some special subclasses $P^l_{n,a}$ of the class $P_{n,a}$ with the gaps. Namely, let $P^l_{n,a}$ denote a class of polynomials of the form $w(z) = 1 + a_l z^l + a_{l+1} z^{l+1} + \ldots + a_n z^n$, $(l \leqslant n)$ which belong to $P_{n,a}$.

**Theorem 6.** *If* $w(z) = 1 + a_l z^l + \ldots + a_n z^n \in P^l_{n,a}$ *then*

$$(21) \qquad |a_k| \leqslant 2(1-a)\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad k = l, l+1, \ldots, n$$

The proof follows by the fact that the extremal polynomials given by (3) are the polynomials with a gap, if $q(z)$ is a polynomial with a gap, too.

Similar theorems can be obtained for the classes $R^l_{n,a}$, $H^l_{n,a}$, $G^l_{n,a}$ of the corresponding polynomials with gaps.

**Remark 3.** Every polynomial of the classes $R_{n,a}$, $H_{n,a}$, $G_{n,a}$ is univalent in $K_1$. All these classes are the subclasses of the class of close-to-convex functions. This follows from the fact that the transformations: $\int_0^z [f(\zeta)/\zeta]d\zeta$ and $2/z\int_0^z f(\zeta)d\zeta$ preserve the class of close-to-convex functions, see for example [3], [4].

MacGregor [6] and others considered the functions $f(z) = z + a_2 z^2 + \ldots$ such that $\operatorname{Re} f(z)/z > 0$ for $z \in K_1$. Here, we consider a similar class of polynomials.

Let $L_{n,a}$ denote a class of polynomials of the form $l(z) = z + a_2 z^2 + \ldots \ldots + a_n z^n$, such that $\operatorname{Re} f(z)/z > a$ for $z \in K_1$.

We can write

$$l(z) \in L_{n,a} \Leftrightarrow l(z) = zw(z) \text{ and } w \in P_{n-1,a}.$$

In this class the coefficients $a_k$ are estimated as follows

$$|a_k| \leqslant 2(1-a)\cos\frac{\pi}{\left[\frac{n-1}{k-1}\right]+2}, \qquad k = 2, 3, \ldots, n.$$

The polynomials of the class $L_{n,a}$ are not necessary univalent in $K_1$. It would be interesting to find the radii of univalence, starlikeness and convexity.

For example, if $n = 2$ then the radii of univalence and starlikeness are equal to $\min\{1, 1/2(1-a)\}$.

Some of these problems were solved in the case $a = 0$ and $n \to \infty$, [6].

## REFERENCES

[1] Egervary E. and Szasz O., *Einige Extremalprobleme in Bereiche der trigonometrischen Polynome*, Math. Z. 27 (1928), 641-652.

[2] Holland F., *Some Extremum Problems for Polynomials with Positive Real Part*, Bull. London Math. Soc., 5 (1973), 54-58.

[3] Lewandowski Z., *On a Problem of M. Biernacki*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, XVII, 5 (1963), 39-41.
[4] Libera R.J., *Some Classes of Regular Univalent Functions*, Proc. Amer. Math. Soc., 16 (1966), 755-758.
[5] ,, , *Univalent a-Spiral Functions*, Can. J. Math., 19 (1967), 449-456.
[6] MacGregor T.H., *Functions whose Derivative has a Positive Real Part*, Trans. Amer. Math. Soc. 104, (1962), 532-537.

## STRESZCZENIE

Niech $P_{n,a}$ oznacza klasę wielomianów postaci $w(z) = 1+a_1z+\ldots$ $\ldots+a_nz^n$ spełniających warunek $\operatorname{Re} w(z) > a$ dla $|z|<1$. W pracy tej oszacowano współczynniki w klasie $P_{n,a}$ oraz w pewnych podklasach wielomianów jednolistnych w kole jednostkowym związanych z tą klasą.

W szczególności, jeżeli $w(z)\in P_{n,a}$ to

$$|a_k| \leqslant 2(1-a)\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad k = 1, 2, \ldots, n.$$

## РЕЗЮМЕ

Пусть $P_{n,a}$ обозначает класс полиномов вида $w(z) = 1+a_1z+\ldots$ $\ldots+a_nz^n$ исполняющих условие $\operatorname{Re} w(z) > a$, где $|z|<1$. В этой работе дано оценки коэффицентов в классе $P_{n,a}$ и в некоторых подклассах однолистных полиномов связянных с классом $P_{n,a}$. В особенности,

если $w(z)\in P_{n,a}$ то $|a_k| \leqslant 2(1-a)\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}$, $K = 1, 2, \ldots, n$.