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On a Certain Class of Regular Functions

O pewnej klasie funkcji regularnych

О некотором классе регулярных функций

1. Let $S_k^*(k \geqslant 1)$ denote the class of all functions of the form

$$f(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \ldots, \tag{1}$$

regular, univalent, and starlike in the disk $K = \{z: |z| < 1\}$. Next, let $S_k^*(M)$ be the class of the functions of the form (1) and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)} - M\right| < M, \quad \text{for every } z \epsilon K, \tag{2}$$

where $M > \frac{1}{2}$ is an arbitrary fixed number.
Evidently

$$S_k^*(M) \subset S_k^*(\infty) \subset S_1^*(\infty) = S_1^*,$$

and

$$S_k^*(M) \subset S_1^*(M) \subset S_1^*(\infty) = S_1^*,$$

where S_1^* is the class of univalent and starlike functions of the form

$$f(z) = z + a_2z^2 + \ldots, \; z \epsilon K.$$

Coefficient estimates for the functions of S_k^* were given by T.H. MacGregor [6]. Some proprietes of the class $S_1^*(M)$ for $M \geqslant 1$ were considered by W. Janowski [5]. In this paper we obtain the coefficient estimates for the functions which belong to $S_k^*(M)$, exact bounds for $|f(z)|$ and $|f'(z)|$, the radius of convexity for $S_k^*(M)$ and some other estimates. The results of W. Janowski [5] and coefficient estimates given by T. H. MacGregor [6] are particular cases of our results.

2. Let $\mathscr{P}$ be the family of all functions of the form

(3) $$P(z) = 1+p_1 z+p_2 z^2+\ldots,$$

regular in the disk K, satisfying the condition

$$\operatorname{Re} P(z) > 0 \text{ for every } z \in K,$$

and let $\mathscr{P}_k$ be the set of functions of the family $\mathscr{P}$ of the form

(4) $$P(z) = 1+p_k z^k+p_{k+1} z^{k+1}+\ldots, \quad k \geqslant 1.$$

Let $\mathscr{P}(M)$ and $\mathscr{P}_k(M)$ be the families of functions (3) and (4) respectively, regular in the disk K and satisfying the condition

(5) $$|P(z)-M| < M \quad \text{for every } z \in K \text{ and } M > \tfrac{1}{2}.$$

First we remark that the condition (5) may be written in the form

(6) $$\left|P(z)-\frac{1}{1-m}\right| < \frac{1}{1-m}, \quad \text{where } m = 1-\frac{1}{M},\ -1 < m \leqslant 1,$$

while the condition (6) is equivalent the following equality

(7) $$P(z) = \frac{1+a}{p(z)+a}, \quad \text{where } p(z) \in \mathscr{P}_k,\ a = \frac{1-m}{1+m}.$$

Denote by Ω_k the family of all functions of the form

(8) $$\omega(z) = c_k z^k+c_{k+1} z^{k+1}+\ldots,$$

regular in K and satisfying the condition $|\omega(z)| < 1$ for $z \in K$. By Schwarz's Lemma we have $|\omega(z)| \leqslant |z|^k$ for every $z \in K$. It is well known that every function of $\mathscr{P}_k$ can be represented in the form

(9) $$p(z) = \frac{1-\omega(z)}{1+\omega(z)}, \quad \text{where } \omega(z) \in \Omega_k.$$

Using this fact, following Lemmas can be proved (the proofs can be obtained in the same way as for $m \geqslant 0$, see [4]).

Lemma 1. *If a function $P(z)$ belongs to the family $\mathscr{P}_k(M)$ then it can be represented in the form*

(10) $$P(z) = \frac{1+\omega(z)}{1-m\omega(z)}, \quad m = 1-\frac{1}{M},$$

where $\omega(z) \in \Omega_k$.

Lemma 2. *If a function* $P(z) \in \mathscr{P}_k(M)$, then for $|z| = r < 1$

$$\frac{1-|z|^k}{1+m|z|^k} \leqslant |P(z)| \leqslant \frac{1+|z|^k}{1-m|z|^k}, \tag{11}$$

and

$$\left|\frac{zP'(z)}{P(z)}\right| \leqslant \frac{k(1+m)|z|^k}{1-(1-m)|z|^k - m|z|^{2k}}. \tag{12}$$

The estimates (11) *and* (12) *are sharp and the signs of equality hold for the function of the form*

$$P(z) = \frac{1+\varepsilon z^k}{1-\varepsilon m z^k}, \quad |\varepsilon| = 1. \tag{13}$$

Lemma 3. *If* $P(z) \in \mathscr{P}_k(M)$, then for $|z| = r < 1$

$$\frac{1-|z|^k}{1+m|z|^k} \leqslant \operatorname{Re} P(z) \leqslant \frac{1+|z|^k}{1-m|z|^k} \tag{14}$$

The bounds are sharp, being attained by (13).

Proof. From (10) we have

$$\operatorname{Re} P(z) = \frac{1+(1-m)\operatorname{Re}\omega(z) - m|\omega(z)|^2}{|1-m\omega(z)|^2} \geqslant \frac{1-|\omega(z)|}{1+m|\omega(z)|} \geqslant \frac{1-|z|^k}{1+m|z|^k}.$$

The proof of the left side is similar.

Now we prove some theorems for the class $\mathscr{P}_1(M) = \mathscr{P}(M)$.

Theorem 1. *If* $P(z) \in \mathscr{P}_1(M)$, *then for* $|z| \leqslant r$

$$|P(z) - H(r, m)| \leqslant G(r, m), \tag{15}$$

where

$$H(r, m) = \frac{(1+a)(a+b)}{a^2+2ab+1}, G(r, m)$$

$$= \frac{R(1+a)}{a^2+2ab+1}, b = \frac{1+r^2}{1-r^2}, R = \frac{2r}{1-r^2}.$$

Proof. From (7) we have

$$p(z) = \frac{1+a}{P(z)} - a.$$

The function $p(z)$ maps the disk $|z| \leqslant r$ on the disk $|p(z)-b| \leqslant R$ and therefore

$$\left|\frac{1}{P(z)}-\frac{a+b}{1+a}\right| \leqslant \frac{R}{1+a};$$

hence

$$\left(\frac{1}{P(z)}-\frac{a+b}{1+a}\right)\left(\frac{1}{\overline{P(z)}}-\frac{a+b}{1+a}\right) \leqslant \frac{R^2}{(1+a)^2}$$

and consequently

$$\left|P(z)-\frac{(1+a)(a+b)}{a^2+2ab+1}\right| \leqslant \frac{R(1+a)}{a^2+2ab+1}.$$

Theorem 1 implies the following results.

Theorem 2. *If $P(z) \in \mathscr{P}_1(M)$, then*

$$|\arg P(z)| \leqslant \arcsin \frac{R}{a+b}, \tag{16}$$

and

Theorem 3. *If $P(z) \in \mathscr{P}_1(M)$, then*

$$|\operatorname{Im} P(z)| \leqslant \frac{R(1+a)}{a^2+2ab+1}. \tag{17}$$

$$\frac{(1+a)(a+b-R)}{a^2+2ab+1} \leqslant \operatorname{Re} P(z) \leqslant \frac{(1+a)(a+b+R)}{a^2+2ab+1} \tag{18}$$

and

$$\frac{(1+a)(a+b-R)}{a^2+2ab+1} \leqslant |P(z)| \leqslant \frac{(1+a)(a+b+R)}{a^2+2ab+1}. \tag{19}$$

The signs of equality in (16)–(19) hold for the function

$$P(z) = \frac{1+\varepsilon z}{1-\varepsilon m z}, \quad |\varepsilon| = 1.$$

For the class $S_1^*(M)$ theorems similar to Theorem 1 through 3 can be proved.

3. For the class $S_k^*(M)$ we have

Theorem 4. *If $f(z) \in S_k^*(M)$, then for $|z| = r$, $0 \leqslant r < 1$, we have*

$$r(1+mr^k)^{-(m+1)/mk} \leqslant |f(z)| \leqslant r(1-mr^k)^{-(m+1)/mk} \quad \textit{for } m \neq 0, \tag{20}$$

$$re^{-r^k/k} \leqslant |f(z)| \leqslant re^{r^k/k} \quad \textit{for } m = 0. \tag{21}$$

Theorem 5. *If $f(z) \in S_k^*(M)$, then for $|z| = r$, $0 \leqslant r < 1$, we have*

$$(1-r^k)(1+mr^k)^{-(mk+m+1)/mk} \leqslant |f'(z)| \leqslant (1+r^k)(1-mr^k)^{-(mk+m+1)/mk} \quad \textit{for } m \neq 0, \tag{22}$$

(23) $$(1-r^k)e^{-r^k/k} \leqslant |f'(z)| \leqslant (1+r^k)e^{r^k/k} \quad \text{for } m=0.$$

Proofs. Since $\dfrac{zf'(z)}{f(z)} \in \mathscr{P}_k(M)$, then by Lemma 3, we have

(24) $$\frac{1-|z|^k}{1+m|z|^k} \leqslant \operatorname{Re} P(z) \leqslant \frac{1+|z|^k}{1-m|z|^k}.$$

Using the equality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = r\cdot\frac{\partial}{\partial r}\log|f(z)|$$

and (24) we find

$$\frac{\partial}{\partial r}\log|f(z)| \leqslant \frac{\partial}{\partial r}\left[\log r - \frac{m+1}{mk}\log(1-mr^k)\right] \quad \text{where } |z|=r.$$

Integrating both sides of this inequality from 0 to r, we obtain

$$|f(z)| \leqslant r(1-mr^k)^{-(m+1)/mk} \quad \text{if } m\neq 0,$$

and

$$|f(z)| \leqslant re^{r^k/k} \quad \text{if } m=0.$$

The proof of the another inequalities are similar.

The inequalities (20)–(23) are sharp and the signs of the equality hold for the function

(25) $$f_\varepsilon(z) = z(1+\varepsilon mz^k)^{-(m+1)/mk}, \quad |\varepsilon|=1; \text{ in (20) and (22)},$$

and

(26) $$f_\varepsilon(z) = z\exp\left(\frac{\varepsilon z^k}{k}\right), \quad |\varepsilon|=1; \text{ in (21) and (23)}.$$

Remark. The Koebe constant for the class $S_k^*(M)$ is equal to

$$(1+m)^{-(m+1)/mk}, \quad m\neq 0.$$

Theorem 6. *The radius of convexity for the family $S_k^*(M)$ is given by the formula*

(27) $$r_c = \sqrt[k]{\frac{2+k(1+m)-\sqrt{k(1+m)[k(1+m)+4]}}{2}}$$

where $m = 1-\dfrac{1}{M}$.

Proof. It is easy to verify that

$$1+\frac{zf''(z)}{f'(z)} = P(z)+\frac{zP'(z)}{P(z)}.$$

Thus

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geqslant \min_{P(z)\in\mathscr{P}_k(M)} \mathrm{Re}\,P(z) - \max_{P(z)\in\mathscr{P}_k(M)} \left|\frac{zP'(z)}{P(z)}\right|, \qquad |z| = r.$$

From (12) and (14), we have

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1-r^k}{1+mr^k} - \frac{k(1+m)r^k}{(1-r^k)(1+mr^k)} = \frac{r^{2k}-[2+k(1+m)]r^k+1}{(1-r^k)(1+mr^k)}. \tag{28}$$

The denominator of the expression on the right-hand side of inequality (28) is positive for $0 \leqslant r < 1$.
Thus the inequality

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$$

is valid for $r = |z| < r_c$, where r_c is given by the formula (27). Hence the radius of convexity for $S_k^*(M)$ is given by the formula (27) and this result is sharp as shown by the functions of the form (25) or (26) ($m = 0$). For $k = 1$, $m = 1$ we obtain $r_c = 2-\sqrt{3}$ [7].

Theorem 7. *If $f(z)\in S_k^*(M)$, then*

$$|a_n| \leqslant \frac{1+m}{n-1} \quad \textit{for } m < 0, \textit{ where } n = k+1, k+2, \ldots, k \geqslant 1, \tag{29}$$

$$|a_n| \leqslant \frac{1}{n-1} \quad \textit{for } m = 0, \textit{ where } n = k+1, k+2, \ldots, k \geqslant 1, \tag{30}$$

$$|a_n| \leqslant \frac{1}{(n-1)(\nu-1)!} \prod_{\mu=0}^{\nu-1} \left(\mu m + \frac{m+1}{k}\right) \quad \textit{for } m = 1-\frac{1}{M} > 0, \tag{31}$$

where $\nu k+1 \leqslant n \leqslant (\nu+1)k$, $\nu = 1, 2, 3, \ldots$.

Proof. The proof is based on a method due to Clunie [1]. If $f(z)\in S_k^*(M)$ then $\dfrac{zf'(z)}{f(z)} \in \mathscr{P}_k(M)$ and

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-m\omega(z)}, \qquad m = 1-\frac{1}{M}, \tag{32}$$

for some function $\omega(z)\in\Omega_k$, where $\omega(z) = c_k z^k + c_{k+1}z^{k+1} + \ldots$ From (32) it follows that

$$zf'(z) - f(z) = \omega(z)[mzf'(z) + f(z)]. \tag{33}$$

By equating coefficients of the power series on both sides of the equation (33), we obtain the relations

$$(n-1)a_n = (1+m)c_{n-1}, \quad n = k+1,\ k+2, \ldots, 2k. \tag{34}$$

Since $|\omega(z)| < 1$, it follows that $\sum_{n=k}^{\infty} |c_n|^2 \leqslant 1$ and therefore

$$\sum_{n=k}^{2k-1} |c_n|^2 \leqslant 1. \tag{35}$$

From (34) and (35) we find that

$$\sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2 \leqslant (1+m)^2. \tag{36}$$

We rewrite (33) as follows

$$\sum_{n=k+1}^{\infty} (n-1)a_n z^n = \omega(z)\Big[(1+m)z + \sum_{n=k+1}^{p-k} (mn+1)a_n z^n\Big] + \sum_{n=p+1}^{\infty} b_n z^n,$$

where the sum $\sum_{n=p+1}^{\infty} b_n z^n$ is convergent in K, and $p \geqslant 2k+1$. This can also be written as

$$\sum_{n=k+1}^{p} (n-1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = \omega(z)\Big[(1+m)z + \sum_{n=k+1}^{p-k} (mn+1)a_n z^n\Big].$$

Put $z = re^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$, $0 < r < 1$; then since $|\omega(z)| < 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Big| \sum_{n=k+1}^{p} (n-1)a_n (re^{i\varphi})^n + \sum_{n=p+1}^{\infty} d_n (re^{i\varphi})^n \Big|^2 d\varphi \leqslant \frac{1}{2\pi}\int_0^{2\pi} \Big|(1+m)re^{i\varphi} + \sum_{n=k+1}^{p-k} (mn+1)a_n (re^{i\varphi})^n\Big|^2 d\varphi$$

Upon integration, we get

$$\sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 r^{2n} + \sum_{n=p+1}^{\infty} |d_n|^2 r^{2n} \leqslant (1+m)^2 r^2 + \sum_{n=k+1}^{p-k} (mn+1)^2 |a^n|^2 r^{2n}. \tag{37}$$

In particular (37) implies

$$\sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 r^{2n} \leqslant (1+m)^2 r^2 + \sum_{n=k+1}^{p-k} (mn+1)^2 |a_n|^2 r^{2n}. \tag{38}$$

Passing to the limit in (38) as $r \to 1$, we conclude that

$$(39) \qquad \sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 \leqslant (1+m)^2 + \sum_{n=k+1}^{p-k} (mn+1)^2 |a_n|^2.$$

For $m < 0$, the inequality (39) implies

$$(p-1)^2 |a_p|^2 \leqslant (1+m)^2 - \Big(\sum_{n=k+1}^{p-k} n(m+1)[n(1-m)-2] |a_n|^2 + \\ + \sum_{n=p-k+1}^{p-1} (n-1)^2 |a_n|^2 \Big).$$

Therefore, it follows that for $n \geqslant 2k+1$, $k \geqslant 1$

$$(40) \qquad |a_n| \leqslant \frac{1+m}{n-1}.$$

The estimate (40) is sharp and the sign of equality hold for the function

$$f_\varepsilon(z) = z(1+\varepsilon m z^{n-1})^{-(m+1)/m(n-1)}, \; m < 0, \; n = k+1, \; k+2, \ldots, \; |\varepsilon| = 1.$$

If $m = 0$, then $|a_n| \leqslant \dfrac{1}{n-1}$ and the extremal function is $f_\varepsilon(z) = z\exp \times \times \left(\dfrac{\varepsilon z^{n-1}}{n-1}\right)$, $|\varepsilon| = 1$. For $m > 0$, the inequality (39) is equivalent to

$$(41) \quad \sum_{n=p-k+1}^{p} (n-1)^2 |a_n|^2 \leqslant (1+m)^2 + (1+m) \sum_{n=k+1}^{p-k} n[n(m-1)+2] |a_n|^2.$$

By an inductive argument we will establish the inequalities

$$(42a) \qquad \sum_{n=\nu k+1}^{(\nu+1)k} (n-1)^2 |a_n|^2 \leqslant \left\{ \frac{k}{(\nu-1)!} \prod_{\mu=0}^{\nu-1} \left(\mu m + \frac{m+1}{k} \right) \right\}^2,$$

$$(42b) \quad \sum_{n=\nu k+1}^{(\nu+1)k} n[n(m-1)+2] |a_n|^2 \leqslant [\nu^2 k^2 (m-1) + 2\nu k m + m + 1] \times \\ \times \left\{ \frac{1}{\nu!} \prod_{\mu=0}^{\nu-1} \left(\mu m + \frac{m+1}{k} \right) \right\}^2.$$

for $\nu = 1, 2, \ldots$.

For $\nu = 1$, (42a) is valid since it is the same as (36).

We can prove (42b) for $\nu = 1$ by using (36) as follows

$$\sum_{n=k+1}^{2k} n[n(m-1)+2]|a_n|^2 = \frac{(m-1)k^2+2km+m+1}{k^2} \times$$

$$\times \sum_{n=k+1}^{2k} \frac{n[n(m-1)+2]k^2}{(m-1)k^2+2km+m+1} |a_n|^2 \leqslant \frac{(m-1)k^2+2km+m+1}{k^2} \times$$

$$\times \sum_{n=k+1}^{2k} (n-1)^2|a_n|^2 \leqslant \{(m-1)k^2+2km+m+1\}\left(\frac{m+1}{k}\right)^2.$$

Now suppose that (42a) and (42b) hold for $\nu = 1, 2 \ldots, q-1$. Using (41) with $p = (q+1)\cdot k$ and the inductive hypothesis concerning (42a), we obtain the inequalities

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2 \leqslant (1+m)^2 + (1+m) \sum_{n=k+1}^{qk} n[n(m-1)+2]|a_n|^2$$

$$= (1+m)^2\left\{1+\frac{1}{1+m}\sum_{\nu=1}^{q-1}\sum_{n=\nu k+1}^{(\nu+1)k} n[n(m-1)+2]|a_n|^2\right\}$$

$$\leqslant (1+m)^2\left\{1+\sum_{\nu=1}^{q-1}\left(\frac{\nu^2k^2(m-1)}{m+1}+\frac{2\nu km}{m+1}+1\right)\left[\frac{1}{\nu!}\prod_{\mu=0}^{\nu-1}\left(\mu m+\frac{m+1}{k}\right)\right]^2\right\}$$

$$= \left\{\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu m+\frac{m+1}{k}\right)\right\}^2.$$

The last equality can be proved with an inductive argument on q. This last sequence of inequalities implies (42a), where $\nu = q$. Continuing our inductive argument, we use (42a) with $\nu = q$ to deduce (42b) for $\nu = q$ as follows.

$$\sum_{n=qk+1}^{(q+1)k} n[n(m-1)+2]|a_n|^2 = \frac{(m-1)q^2k^2+2qkm+m+1}{(qk)^2} \sum_{n=qk+1}^{(q+1)k} \times$$

$$\frac{n[n(m-1)+2](qk)^2}{(m-1)q^2k^2+2qkm+m+1}|a_n|^2 \leqslant \frac{(m-1)q^2k^2+2qkm+m+1}{(qk)^2} \times \sum_{n=qk+1}^{(q+1)k} \times$$

$$\times (n-1)^2|a_n|^2 \leqslant \frac{(m-1)q^2k^2+2qkm+m+1}{(qk)^2}\cdot\left\{\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu m+\frac{m+1}{k}\right)\right\}^2$$

$$= \{(m-1)q^2k^2+2qkm+m+1\}\cdot\left\{\frac{1}{q!}\prod_{\mu=0}^{q-1}\left(\mu m+\frac{m+1}{k}\right)\right\}^2.$$

We used above the inequality

$$\frac{n[n(m-1)+2]}{(m-1)q^2k^2+2qkm+m+1} \leqslant \left(\frac{n-1}{qk}\right)^2$$

which is valid for positive integer q, $0 < m \leqslant 1$ and $n = q\cdot k+1, \ldots,$ $n = (q+1)k$, $k = 1, 2, \ldots$.

This completes the prof of (42a) and (42b).

Theorem 7 now follows from (42a).

The estimate for $|a_n|$ is precise if n is of the form $n = \nu k+1$, and equality hold only for function (25), i. e

$$f_\varepsilon(z) = z(1+\varepsilon m z^k)^{-(m+1)/mk}, \quad |\varepsilon| = 1.$$

Of course this function belong to $S_k^*(M)$, because

$$\left|\frac{zf'_\varepsilon(z)}{f_\varepsilon(z)} - M\right| = \left|\frac{1-\varepsilon z^k}{1+\varepsilon m z^k} - \frac{1}{1-m}\right| = \left|\frac{-m-\varepsilon z^k}{1+\varepsilon m z^k}\cdot\frac{1}{1-m}\right| < \frac{1}{1-m} = M.$$

We observe that

$$f_\varepsilon(z) = z + \binom{-a}{1}(\varepsilon m)z^{k+1} + \binom{-a}{2}(\varepsilon m)^2 z^{k+2} + \ldots + \binom{-a}{\nu}(\varepsilon m)^\nu z^{\nu k+1} + \ldots,$$

and so

$$a_{\nu k+1} = \binom{-a}{\nu}(\varepsilon m)^\nu, \quad \text{where } a = \frac{m+1}{mk} \text{ i.e.}$$

$$a_{\nu k+1} = \frac{(-1)^\nu \varepsilon^\nu}{\nu!}\prod_{\mu=0}^{\nu-1}\left(\mu m + \frac{m+1}{k}\right).$$

If $k+1 < n < 2k+1$ the estimate for $|a_n|$ is sharp for the some function where k is replaced by n. For all other values of n, theorem 7 does not give sharp bounds.

Remark. If $k = 1$ we obtain the result of W. Janowski [5]. In the case $M = \infty$, i. e. $m = 1$ the above results are the coefficient estimates of T. H. MacGregor [6]. It can be observed that for $k = 2$, $M = \infty$ this theorem gives the well known result of Golusin [2], which asserts that if $f(z) = z + a_2 z^2 + \ldots$, and $a_2 = 0$ then the estimate $|a_n| \leqslant 1$, hold for $n = 3, 4, 5, \ldots$.

Theorem 8. *If $f(z)$ and $g(z)$ belong to the class $S_k^*(M)$ then*

$$F_\lambda(z) = \lambda f(z) + (1-\lambda)g(z) \in S_k^*(M), \quad (0 \leqslant \lambda \leqslant 1).$$

Proof.

$$\left|\frac{zF_\lambda(z)}{F_\lambda(z)} - M\right| \leqslant \lambda M\left|\frac{zf'(z)}{Mf(z)} - 1\right| + (1-\lambda)M\left|\frac{zg'(z)}{Mg(z)} - 1\right| < \lambda M + (1-\lambda)M = M.$$

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STRESZCZENIE

Tematem pracy są pewne zagadnienia ekstremalne w rodzinach $\mathscr{P}_k(M)$ i $S_k^*(M)$, ($M > \frac{1}{2}$, $k = 1, 2, \ldots$,) funkcji regularnych w kole $K = \{z: :|z| < 1\}$ postaci

$$P(z) = 1 + p_k z^k + p_{k+1} z^{k+1} + \ldots$$

lub

$$f(z) = z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \ldots$$

spełniających w tym kole odpowiednio warunek:

$$|P(z) - M| < M \quad \text{lub} \quad \left|\frac{zf'(z)}{f(z)} - M\right| < M \text{ dla } z \in K.$$

Podano dokładne oszacowania na współczynniki, moduł funkcji i moduł pochodnej oraz promień wypukłości w klasie $S_k^*(M)$, jak też pewne oszacowania dla klas $\mathscr{P}_k(M)$ i $\mathscr{P}_1(M)$. Otrzymane wyniki stanowią uogólnienie rezultatów W. Janowskiego [5] i T. H. MacGregora [6].

РЕЗЮМЕ

В работе рассмотрены некоторые экстремальные вопросы в классах $\mathscr{P}_k(M)$ и $S_k^*(M)$, ($M > 1/2$, $k = 1, 2 \ldots$,) регулярных функций в круге

$K = \{z: |z| < 1\}$ вида

$$P(z) = 1 + p_k z^k + p_{k+1} z^{k+1} + \dots$$

или

$$f(z) = z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \dots ,$$

которые удовлетворяют в этом круге соответственно условие $|P(z) - M| < M$ или $|zf'(z)/f(z) - M| < M$ для $z \in K$. Даны точные оценки для коэффициентов, модуль функции, модуль производной и радиус выпуклости для класса $S_k^*(M)$, а также некоторые оценки для классов $\mathcal{P}_k(M)$ и $\mathcal{P}_1(M)$. Полученные результаты обобщают результаты В. Яновского [5] и Т. Х. МакГрегора [6].