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# Some Extremal Problems for the Class $S_a$

Pewne problemy ekstremalne dla klasy $S_a$

Некоторые экстремальные проблемы для класса $S_a$

## 1. On a connection between the class $S_a$ and the class $P$ of functions of positive real part

Let $S_a$, $S_\gamma^*$, $0 \leqslant a, \gamma \leqslant 1$ denote classes of functions of the form

(1) $$f(z) = z + a_2 z^2 + \ldots$$

that are holomorphic in the disc $K_1$ ($K_r = \{z : |z| < r\}$) and satisfy the conditions

(2) $$\left|\arg \frac{zf'(z)}{f(z)}\right| < a\pi/2 \quad \text{for } z \in K_1,$$

(2′) $$\operatorname{re} \frac{zf'(z)}{f(z)} > \gamma \quad \text{for } z \in K_1,$$

respectively.

Let $P$ denote the class of functions

(3) $$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

holomorphic in $K_1$ and having the positive real part there.

In this paper using a connection between $S_a$ and $P$ we obtain estimates of some functionals in the class $S_a$. We obtain also some relations between $S_a$ and $S_\gamma^*$.

It follows from the definition of the class $S_a$ that if $f \in S$ than $p(z) = [zf'(z)/f(z)]^{1/a} \in P$. Conversly if $p \in P$ and a function $f(z) = z + a_2 z^2 + \ldots$ is defined by the formula $zf'(z)/f(z) = p^a(z)$ then $f \in S_a$.

From this we obtain a structural formula of the class $S_a$.

**Theorem 1.** *A function $f$ analytic in $K_1$ belongs to the class $S\alpha$ if and only if there exists a function $p \in P$ such that*

$$f(z) = z\exp\int_0^z \frac{p^\alpha(\zeta)-1}{\zeta}d\zeta \tag{4}$$

*holds,*

**Proof.** A function $f \in S$ if and only there exists a function $p \in P$ such

$$zf'(z)/f(z) = p^\alpha(z) \tag{5}$$

and (4) follows by integrating.

## 2. Some estimates in the class $S_\alpha$

From (5) we obtain the following

**Theorem 2.** *If $f \in S_\alpha$, then we have for $|z| = r < 1$*

$$\left(\frac{1-r}{1+r}\right)^\alpha \leqslant \operatorname{re}\frac{zf'(z)}{f(z)} \leqslant \left(\frac{1+r}{1-r}\right)^\alpha, \tag{6}$$

$$\left(\frac{1-r}{1+r}\right)^\alpha \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \left(\frac{1+r}{1-r}\right)^\alpha, \tag{7}$$

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant 2\alpha\operatorname{arc\,tg} r, \tag{8}$$

*and these estimates are sharp. The extremal functions have the form*

$$F(z) = z\exp\int_0^z \left[\left(\frac{1+\varepsilon\zeta}{1-\varepsilon\zeta}\right)^\alpha - 1\right]\frac{d\zeta}{\zeta}, \quad |\varepsilon| = 1, \tag{9}$$

*and map the disc $K_1$ onto the domains bounded by two logarythmic spirals joining the point $\bar{\varepsilon} r_\alpha$ to the point $-\bar{\varepsilon} r_\alpha \exp\left\{\pi \operatorname{tg}\frac{\alpha\pi}{2}\right\}$, where $r_\alpha$ is the radius of the largest disc centered at the origin and covered by values of functions of the class $S_\alpha$[3].*

Proof. In order to obtain (6) and (7) it is sufficient to consider the formula (5) and well-known estimates of $\operatorname{re} p(z)$ and $|p(z)|$ in the class $P$ whereas (8) follows from the fact $\arg p^\alpha(z) = \alpha \arg p(z)$. The extremal function $F(z)$ corresponds to the function $p(z) = (1+\varepsilon z)/(1-\varepsilon z)$, $|\varepsilon| = 1$, by the formula (5).

Using the structural formula (4) one can easily establish the following

**Theorem 3.** *If $f \in S_\alpha$ and $|z| = r < 1$, then*

$$r\exp\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\binom{-\alpha}{n-k}\frac{r^n}{n} \leqslant |f(z)| \tag{10}$$

$$\leqslant r\exp\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{-\alpha}{k}\binom{\alpha}{n-k}\frac{r^n}{n},$$

$$\left(\frac{1-r}{1+r}\right)^{a} \exp \sum_{n=1}^{\infty} \sum_{k=0}^{n} (-1)^{k} \binom{a}{k} \binom{-a}{n-k} \frac{r^{n}}{n} \leqslant |f'(z)| \tag{11}$$

$$\leqslant \left(\frac{1-r}{1+r}\right)^{a} \exp \sum_{n=1}^{\infty} \sum_{k=0}^{n} (-1)^{k} \binom{-a}{k} \binom{a}{n-k} \frac{r^{n}}{n}.$$

*and both results are sharp. The extremal functions have the form* (9).

## 3. The radius of strong starlikness of order $a$ and some other radii

We shall start with some notations that we will need further.

Let us denote by $S$, $S^*$, $S^*_\beta$ families of functions as follows:

(i) $f \in S \Leftrightarrow f(z) = z + a_2 z^2 + \ldots$ regular and univalent in $K_1$,

(ii) $f \in S^* \Leftrightarrow f \in S$ and $\operatorname{re}\{zf'(z)/f(z)\} > 0$, $z \in K_1$,

(iii) $f \in S^*_\beta \Leftrightarrow f \in S$ and $\operatorname{re}\{zf'(z)/f(z)\} > \beta$, $0 \leqslant \beta < 1$, $z \in K_1$.

Let us observe that $S^* = S^*_0 = S_1$.

A function $f(z) = z + a_2 z^2 + \ldots$ shall be called starlike (starlike of order $\beta$; strongly starlike of order $a$) in the disc $K_\varrho$ if and only if $\operatorname{re}\{zf'(z)/f(z)\} > 0$ ($\operatorname{re}\{zf'(z)/f(z)\} > \beta$, $|\arg\{zf'(z)/f(z)\}| < a\pi/2$) holds in $K_\varrho$.

Connections between univalence, the strong starlikness of order $a$ and starlikeness of order $\beta$ are given by the following theorems.

**Theorem 4.** *Each function $f \in S$ is a strongly starlike function of order $a$ in the disc $K_{R(a)}$, where*

$$R(a) = \operatorname{th} \frac{a\pi}{4}. \tag{12}$$

**Proof.** If $f \in S$ then [c f. 2]

$$|\arg\{zf'(z)/f(z)\}| \leqslant \log \frac{1+|z|}{1-|z|}.$$

holds for $|z| < 1$. It follows that a function $f$ is strongly starlikness of order $a$ in a disc $K_R$ if

$$\log \frac{1+|z|}{1-|z|} < \frac{a\pi}{2} \quad \text{for } |z| < R.$$

Hence the number $R(a)$ is the smallest positive root of the equation

$$\log \frac{1+R}{1-R} = \frac{a\pi}{2}.$$

The equation has exactly one root in the interval (0, 1) given by the formula (12). The theorem has been established.

**Corollary 1.** *Taking $a = 1$ we get the radius of starlikeness of the class $S$,*

$$R(1) = r^* = \mathrm{th}\frac{\pi}{4}. \tag{13}$$

**Theorem 5.** *Each function $f$ of the class $S_\beta^*$ is strongly starlike of order $a$ in the disc $|z| < R^*(\beta, a)$ where*

$$R^*(\beta,a) = \begin{cases} \dfrac{1-\beta-\sqrt{(1-\beta)^2-(1-2\beta)\sin^2(a\pi/2)}}{(1-2\beta)\sin(a\pi/2)} & \text{for } \beta \neq 1/2 \\ \sin(a\pi/2), & \text{for } \beta = 1/2. \end{cases} \tag{14}$$

*The number $R^*(\beta, a)$ is the best possible one. The extremal functions have the form*

$$F_\beta(z) = z(1-\varepsilon z)^{-2(1-\beta)}, \quad |\varepsilon| = 1. \tag{15}$$

**Proof.** If $f \in S_\beta^*$ then there exists a function $p \in P$ such that

$$zf'(z)/f(z) = (1-\beta)p(z)+\beta.$$

Hence if $f \in S_\beta^*$ then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant \arcsin\frac{2r}{1+r^2+\dfrac{\beta}{1-\beta}(1-r^2)} \tag{16}$$

holds for $|z| = r < 1$ [1].

Hence, a function $f$ is strongly starlike of order $a$ in the disc $K_r$ if the inequality

$$\arcsin\frac{2r}{1+r^2+\dfrac{\beta}{1-\beta}(1-r^2)} \leqslant a\pi/2, \tag{17}$$

or an equivalent one

$$\frac{2r}{1+r^2+\dfrac{\beta}{1-\beta}(1-r^2)} \leqslant \sin(a\pi/2) \tag{18}$$

holds.
Taking the sign of equality in (18) we obtain the equation for $R^*(\beta, a)$ whose smallest positive root is given by (14).

The proof of the first part of Theorem 5 has been established.

The equality in (16) occurs for the functions $F$ given by (15), and these functions are the extremal ones in this problem. They are strongly starlike of order $a$ in the disc of the radius $R^*(\beta, a)$ and only in this disc.

**Corollary 2.** *If $f \in S^c$, $S^c$ is the class of convex functions, then $f$ is strongly starlike of order $\alpha$, at least in the disc $|z| < R^c_\beta = \sin(\alpha\pi/2)$. The number $R^c_\beta$ is exact one. The extremal functions have the form*

$$F(z) = F_{1/2}\ (z) = z/(1-\varepsilon z), \quad |\varepsilon| = 1.$$

**Proof.** It is well-known that $S^c \subset S_{1/2}$ and that the function $F_\beta$ given by (15) is convex for $\beta = 1/2$. Taking in account these two remarks and Theorem 5 we obtain Corollary 2.

Taking $\beta = 0$ in Theorem 5 we obtain

**Corollary 3.** *Each function of the class $S^*$ is strongly starlike of order $\alpha$, at least, in the disc*

$$|z| < R^*(0, \alpha) = \operatorname{tg}(\alpha\pi/4).$$

*The extremal functions is the Koebe function $F(z) = z/(1-\varepsilon z)^2$, $|\varepsilon| = 1$.*

**Theorem 6.** *Each function of the class $S_\alpha$ is starlike of order $\beta$, at least, in the disc*

$$|z| < R_*(\alpha, \beta) = (1-\beta^{1/2})/(1+\beta)^{1/2}.$$

*The number $R^*(\alpha, \beta)$ is the best possible one. The extremal functions are of the form* (9).

**Proof.** In view of (6) a function $f \in S_\alpha$ is starlike of order $\beta$ if

$$\left(\frac{1-r}{1+r}\right)^\alpha \geqslant \beta.$$

Hence, $R_*(\alpha, \beta)$ is given by the equation

$$\left(\frac{1-r}{1+r}\right)^\alpha = \beta.$$

The example of the function (9) shows that the number $R_*(\alpha, \beta)$ can not be improved since functions (9) are starlike of order $\beta$ if and only if $|z| < R_*(\alpha, \beta)$.

**Corollary 4.** *If $f \in S^*$ then it is starlike of order $\beta$ at least, in the disc*

$$|z| < R_*(1, \beta) = (1-\beta)/(1+\beta).$$

*In particular, it is starlike of order $1/2$ in the disc*

$$|z| < 1/3.$$

**Remark.** Let us observe that $R^*(\beta, \alpha)$, $R_*(\beta, \alpha)$ may take the value 1 only in the limit cases $\beta = 0$, $\alpha = 1$ or $\beta = 1$, $\alpha = 0$. It shows that the classes $S^*_\beta$ and $S_\alpha$ don't include each other for $0 < \beta < 1$ and $0 < \alpha < 1$. These classes are always different ones.

**Theorem 7.** *If $f \in S_\gamma$ then it is strongly starlike of order $a$, at least, in the disc $|z| < R(\gamma, a)$, where*

$$R(\gamma, a) = \begin{cases} \operatorname{tg}(a\pi/4\gamma) & \text{for } a < \gamma, \\ 1 & \text{for } a \geqslant \gamma. \end{cases}$$

*The radius $R(\gamma, a)$ cannot be improved. The extremal functions have the form* (9).

**Proof.** If $f \in S_\gamma$ then, in view of (8), we have

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqslant 2\gamma \operatorname{arc tg}|z|.$$

Hence a function $f$ is strongly starlike of order $a$ if the condition

$$2\gamma \operatorname{arc tg}|z| < a\pi/2$$

holds. It follows that $R(\gamma, a)$ is the smallest positive root of the equation

$$2\gamma \operatorname{arc tg} r = a\pi/2.$$

If $a \geqslant \gamma$ then $S_a \subset S_\gamma$.
Theorem 7 has been established.

**Theorem 8.** *Each function of the class $S_a$ is convex, at least, in the disc $|z| < R_c^{(a)}$ where $R_c(a)$ is the smallest positive root of*

$$(1-r)^{1+a}(1+r)^{1-a} - 2ar = 0.$$

*The number $R_c(a)$ cannot be improved. The extremal functions have the form* (9).

**Proof.** From (5) we obtain

$$\frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} = a\frac{p'(z)}{p(z)}.$$

In view of (5) after some calculations we have

$$1 + \frac{zf''(z)}{f'(z)} = p^a(z) + a\frac{zp'(z)}{p(z)}.$$

Thus

$$\operatorname{re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \operatorname{re} p^a(z) - a\left|\frac{zp'(z)}{p(z)}\right| \geqslant \left(\frac{1-r}{1+r}\right)^a - a\frac{2r}{1-r^2}.$$

Since $f(z)$ has to be convex in the disc $|z| < R$ so the condition

$$\left(\frac{1-r}{1+r}\right)^a - a\frac{2r}{1-r^2} > 0$$

must be satisfied for $r < R$.

Hence we obtain Theorem 8.

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## STRESZCZENIE

W pracy tej rozważana jest klasa $S_a$ funkcji $a$-kątowo gwiaździstych określona warunkiem (2). Korzystając ze związku pomiędzy klasą $S_a$ a klasą $P$ funkcji o części rzeczywistej dodatniej wyprowadzono oszacowania pewnych funkcjonałów (6), (7), (8), (10), (11) w klasie $S_a$.

W dalszej części pracy wyliczone zostały promienie $a$-kątowej gwiaździstości w różnych klasach funkcji analitycznych i na odwrót różne promienie w klasie $S_a$.

## РЕЗЮМЕ

В работе рассмотрен класс $S_a$, $a$-углово-звездных функций, который определяется условием (2). Используя связь между классом $S_a$ и классом $P$ функций с реальной положительной частью, выведены оценки некоторых функционалов (6), (7), (8), (10), (11) в классе $S_a$.

Кроме того вычислены радиусы $a$-угловой звездности в разных классах аналитических функций и наоборот — разные радиусы в классе $S_a$.