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The Radius of Convexity and Starlikeness for Certain Classes of Analytic Functions with Fixed Second Coefficients

Promień wypukłości i gwiaździstości dla pewnych klas funkcji analitycznych z ustalonym drugim współczynnikiem

Радиус выпуклости и звездности для некоторых классов аналитических функций с фиксированным вторым коэффициентом

1. Introduction

Let S^* denote the class of functions

$$f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$$

which are regular, univalent and starlike in the unit disc $D: |z| < 1$. It is well known that the radius of convexity of the above class of functions is $2-\sqrt{3}$.

In this paper we shall generalize this result and find the radius of convexity of the class S of starlike functions by fixing the value of a_2. Without loss of generality we may assume that a_2 is non-negative since $e^{-i\theta} \cdot f(ze^{i\theta})$ belongs to S whenever $f(z)$ belongs to S.

We shall further consider the class of analytic functions

$$f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$$

which satisfy the condition

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0 \text{ for } |z| < 1.$$

Mac Gregor has proved that the radius of starlikeness of this class is $\sqrt{2}-1$, [1]. We shall generalize this result and determine the radius of starlikeness when a_2 is fixed.

2. Basic lemma (*)

We shall need the following

Lemma. *If*

$$P(z) = 1 + b_2 z + \ldots + b_{n+1} z^n + \ldots$$

is regular in the unit disc D and has positive real part for $|z| < 1$ *then*

(1) $$\left| \frac{zP'(z)}{P(z)} \right| \leqslant \frac{2r}{1-r^2} \cdot \frac{pr^2 + 2r + p}{r^2 + 2pr + 1},$$

where $r = |z|$, $p = \frac{1}{2}|b_2|$.

Proof. We can as before assume b_2 to be non-negative. Put $b_2 = 2p$ so that $0 \leqslant p \leqslant 1$.

Since real part of $P(z)$ is positive for $|z| < 1$, therefore

(2) $$P(z) = \frac{1 + g(z)}{1 - g(z)},$$

where

$$g(z) = pz + \ldots$$

and

$$|g(z)| < 1 \text{ for } |z| < 1.$$

Differentiating (2) we get

(3) $$\frac{zP'(z)}{P(z)} = \frac{2zg'(z)}{1 - g((z))}.$$

Let

(4) $$h(z) = \frac{\frac{g(z)}{z} - p}{1 - p \cdot \frac{g(z)}{z}},$$

then $h(0) = 0$ and $|h(z)| < 1$. Therefore by Schwarz's Lemma $|h(z)| \leqslant |z|$. From (4) we have

(5) $$g(z) = \frac{z(p + h(z))}{1 + ph(z)}.$$

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*As pointed out to the author by the referee, the same result as in Lemma 1 and Theorem 1 has been obtained independently by: D. E. Tepper – T. A. M. S. vol. 150, No 2, August 1970, p. 519–529 ("On the radius of convexity and boundary distortion of schlicht functions").

Differentiation and simplication gives

$$\frac{g'(z)}{1-(g(z))^2} = \frac{\dfrac{p+h(z)}{1+ph(z)} + (1-p^2)\dfrac{zh'(z)}{(1+ph(z)^2)}}{1-z^2\left(\dfrac{p+h(z)}{1+ph(z)}\right)_2}.$$

Putting

$$\frac{p+h(z)}{1+ph(z)} = h_1(z), \quad \text{then} \quad \frac{(1-p^2)h'(z)}{(1+ph(z))^2} = h_1'(z).$$

Therefore

(6) $$\frac{g'(z)}{1-(g(z))^2} = \frac{h_1(z)+zh_1'(z)}{1-z^2(h_1(z))^2}.$$

Obviously $h_1(z)$ is analytic in D and $|h_1(z)|<1$, therefore

(7) $$|h_1'(z)| \leqslant \frac{1-|h_1(z)|^2}{1-|z|^2}.$$

Using (7) we get from (6)

(8) $$\left|\frac{g'(z)}{1-(g(z))^2}\right| \leqslant \frac{|h_1(z)|+|z|\dfrac{1-|h_1(z)|^2}{1-|z|^2}}{1-|z|^2|h_1(z)|^2}$$

$$= \frac{(|h_1(z)|+|z|)(1-|z|\,|h_1(z)|)}{(1-|z|^2)(1+|z|\,|h_1(z)|)(1-|z|\,|h_1(z)|)} = \frac{|h_1(z)|+|z|}{(1-|z|^2)(1+|z|\,|h_1(z)|)}.$$

Since $|h(z)| \leqslant |z|$, therefore by [2]

(9) $$|h_1(z)| \leqslant \frac{p+|z|}{1+p|z|}.$$

The inequality (8) gives in conjunction with (9)

(10) $$\left|\frac{g'(z)}{1-(g(z))^2}\right| \leqslant \frac{pr^2+2r+p}{(1-r^2)(r^2+2pr+1)}, \qquad r=|z|.$$

From (3) and (10) we get

$$\left|\frac{zP'(z)}{P(z)}\right| \leqslant \frac{2r}{1-r^2}\cdot\frac{pr^2+2r+p}{r^2+2pr+1}.$$

The function $P(z) = \dfrac{1-z^2}{1-2pz+z^2}$ shows that the result is sharp and the upper bound is attained for $z=-r$.

3. Radius of convexity for starlike functions

Theorem 1. *Let* $f(z) = z + 2pz^2 + \dots$ *be regular, univalent and starlike in the unit disc,* $0 \leqslant p \leqslant 1$. *Then* $f(z)$ *is convex for* $|z| < r_0$, *where* r_0 *is the smallest positive root of the equation*

$$1 - 2pr - 6r^2 - 2pr^3 + r^4 = 0. \tag{11}$$

The result is sharp.

Proof. Since $f(z)$ is starlike in $|z| < 1$, therefore

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad \text{for } |z| < 1.$$

Let

$$\frac{zf'(z)}{f(z)} = P(z), \tag{12}$$

then $P(0) = 1$ and $\operatorname{Re}[P(z)] > 0$ for $|z| < 1$ and

$$P(z) = 1 + 2pz + \dots$$

Differentiating (12) we get after some simplification

$$1 + \frac{zf''(z)}{f'(z)} = P(z) + \frac{zP'(z)}{P(z)}.$$

we have

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geqslant \operatorname{Re} P(z) - \left|\frac{zP'(z)}{P(z)}\right|$$

$$\geqslant \frac{1 - r^2}{1 + 2pr + r^2} - \frac{2r}{1 - r^2} \cdot \frac{pr^2 + 2r + p}{r^2 + 2pr + 1},$$

where we have used the estimate given in [3], p. 393:

$$\operatorname{Re} P(z) \geqslant \frac{1 - r^2}{1 + 2pr + r^2}.$$

Therefore, we get

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geqslant \frac{1 - 2pr - 6r^2 - 2pr^3 + r^4}{(1 - r^2)(1 + 2pr + r^2)}.$$

From this result we obtain the radius of convexity as given above.

We shall show that the result obtained is sharp.

Let

$$f(z) = \frac{z}{1-2pz+z^2}$$

then $f(z)$ is obviously starlike and for this function we have

$$1+\frac{zf''(z)}{f'(z)} = \frac{1+2pz-6z^2+2pz^3+z^4}{(1-z^2)(1-2pz+z^2)} = 0 \quad \text{when } z = -r_0.$$

It follows that $f(z)$ is not convex in any larger circle.

This completes the proof of the theorem.

Corollary 1.1. *When $p = 1$, the above equation* (11) *reduces to $1-4r+$ $+r^2 = 0$ and we obtain the radius of convexity $r_0 = 2-\sqrt{3}$.*

Corollary 1.2. *Let $f(z) = z+a_3z^3+a_5z^5+\ldots$ be an odd starlike function. Then the radius of convexity $r_0 = [3-2\sqrt{2}]^{1/2}$.*

Proof. An odd function has vanishing even coefficients and therefore on putting $p = 0$ in (11) we get the required result.

4. Radius of starlikeness

Theorem 2. *If $f(z) = z+2pz^2+,\ldots$ is analytic in D and satisfies the condition*

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0,$$

then $f(z)$ is univalent and starlike for

(13) $$|z| < r_0,$$

where r_0 is the smallest positive root of the equation

(14) $$1-4r^2-4pr^3-r^4 = 0.$$

The result is sharp.

Proof. Since

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0 \quad \text{for } |z| < 1,$$

therefore

(15) $$\frac{f(z)}{z} = P(z),$$

where $P(0) = 1$, $\mathrm{Re}\,P(z) > 0$ for $|z| < 1$ and

$$P(z) = 1+2pz+\ldots$$

Differentiating (15), we get

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zP'(z)}{P(z)}.$$

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqslant 1 - \left|\frac{zP'(z)}{P(z)}\right| \geqslant 1 - \frac{2r}{1-r^2} \cdot \frac{pr^2+2r+p}{r^2+2pr+1}$$

$$= \frac{1-4r^2-4pr^3-r^4}{(1-r^2)(r^2+2pr+1)}.$$

$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$ if $1-4r^2-4pr^3-r^4 > 0$. Therefore the radius of starlikeness is given by the smallest positive root r_0 of the equation

$$1-4r^2-4pr^3-r^4 = 0.$$

For the function $f(z) = \frac{z(1-z^2)}{1-2pz+z^2}$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1-4z^2+4pz^3-z^4}{(1-z^2)(z^2-2pz+1)},$$

which vanishes for $z = -r_0$.

This shows that the result obtained is sharp.

Corollary 2.1 *When* $p = 1$, $r_0 = \sqrt{2}-1$, *which is the result obtained by MacGregor in* [1].

Corollary 2.2 *Let* $f(z) = z + a_3z^3 + \ldots$ *be an odd function such that* $\operatorname{Re}\left|\frac{f(z)}{z}\right| > 0$, *then* $f(z)$ *is univalent and starlike for* $|z| < (\sqrt{5}-2)^{1/2}$.

Proof. Since an odd function has vanishing even coefficients, therefore the result follows by taking $p = 0$ in (14).

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REFERENCES

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[3] Robertson, M. S., *On the Theory of Univalent Functions*, Ann. of Math., 37 (1936), 374–408.

STRESZCZENIE

Niech $f(z) = z + a_2 z^2 + \ldots$ będzie funkcją holomorficzną w kole jednostkowym. Autor rozważa dwie klasy funkcji o powyższym rozwinięciu:

(1) funkcje spełniające warunek $\mathrm{Re}[zf'(z)/f(z)] > 0$ z ustalonym a_2,

(2) funkcje spełniające warunek $\mathrm{Re}[f(z)/z] > 0$ z ustalonym a_2. W klasie (1) podano dokładną wartość promienia wypukłości, a w klasie (2) promienia gwiaździstości oraz pewne wnioski wynikające z tych wyników. Otrzymane rezultaty uogólniają między innymi wynik MacGregora [1].

РЕЗЮМЕ

Пусть $f(z) = z + a_2 z^2 + \ldots$ — голоморфная функция в единичном круге. Автор исследует два класса функций с вышеуказанным разложением:

(1) функции, которые удовлетворяют условие $\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$ при фиксированном a_2,

(2) функции, которые удовлетворяют условие $\mathrm{Re}[f(z)/z] > 0$ при фиксированном a_2.

Даны точные границы выпуклости для класса (1) и звездности для класса (2), а также некоторые последствия этих результатов. Результаты автора обобщают, в частности, результат Т. Х. МакГрегора [1].