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The Region of Variability of the Ratio $f(b)/f(c)$ within the Class of Meromorphic and Univalent Functions in the Unit Disc.

Obszar zmienności stosunku $f(b)/f(c)$ w klasie funkcji meromorficznych i jednolistnych w kole jednostkowym

Область значений выражения $f(b)/f(c)$ в классе мероморфных и однолистных функций в единичном круге

1. Introduction.

Let U_p denote the class of functions meromorphic and univalent in the unit disc Δ subject to the conditions

$$f(0) = 0, f'(0) = 1, f(p) = \infty \tag{1.1}$$

where p is fixed, $0 < p < 1$.

Let $\mathcal{M}_p$ be the family of functions meromorphic and univalent in Δ and satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad f(0) = 0 \\ &\text{(ii)} \quad f(z_0) = z_0,\ z_0 \neq 0,\ z_0 \in \Delta \\ &\text{(iii)} \quad f(p) = \infty,\ p \neq z_0 \end{aligned} \tag{1.2}$$

Various problems concerning functions that are holomorphic and univalent in Δ and are normalized by the conditions (i) and (ii) in (1.2) have been considered by many authors. In particular, J. Krzyż [3] found the region of variability of $\varphi(z)$ for fixed $z \in \Delta$, where φ ranges over the whole class of functions that satisfy (i) and (ii) in (1.2).

In the present note we determine the region of variability of $f(z)$, $f \in \mathcal{M}_p$. In the limit case $p = 1$ we obtain the Krzyż's result.

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2. Preliminary remarks

Let $z, z \neq 0, p$, be a fixed point of Δ and let the variability region of $\varphi(z)$ over $\mathscr{M}_p$ be the set $E = \{w\colon w = \varphi(z), \varphi \in \mathscr{M}_p\}$.

It is clear that E is identical to the set of all possible values of the ratio $z_0 f(z)/zf(z_0)$ for f ranging over the whole class U_p. The set E is closed because the class U_p is compact.

Let ∂E denote a boundary of E and let $\mathscr{C}E$ be the complement of E. A point $P \in \partial E$ is said to be a non-singular boundary point of the set E if there exists a point $a, a \in \mathscr{C}E$, such that $|P-a|, P \in E$, attains its minimum with $P = P_0$. It is well-known [5] that the set of non-singular boundary points is everywhere dense in ∂E.

Functions $f \in U_p$ corresponding to the non-singular boundary points of Δ we shall call extremal functions.

In order to determine the set of non-singular boundary points we shall use variational formulas given by the following:

Theorem A. [4]. *Suppose that $f \in \mathrm{U}_p$, $\zeta, \zeta \neq p$, is a fixed point of Δ, A is an arbitrary fixed complex number, ζ_0 satisfies $|\zeta_0| = 1$ and $a = -p\,\mathrm{res}_{z=p} f(z)$. Then there exists a positive number λ_0 such that for each $\lambda \in \langle 0, \lambda_0)$ there exist functions belonging to U_p that have the form*

(2.1)
$$f^*(z) = f(z) - \lambda\left\{A\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \frac{2f^2(z)}{f(\zeta)-f(z)} + AP(z,\zeta) + \bar{A}P(z,\bar{\zeta}^{-1})\right\} + O(\lambda^2),$$

$$f^2(z) = f(z) + \lambda P(z,\zeta_0) + O(\lambda^2) \tag{2.2}$$

where

$$P(z,u) = f(z) - zf'(z)\frac{u+z}{u-z} + af^2(z)\frac{u+p}{u-p}. \tag{2.3}$$

If w_0 is an interior point of $\mathscr{C}f(\Delta)$, then

$$f^{**}(z) = f(z) - \lambda A\frac{f^2(z)}{w_0 - f(z)} + O(\lambda^2) \tag{2.4}$$

belongs to U_p.

3. A differential equation for the extremal functions

Let $b, c, b \neq c, \neq 0 \neq p$ be fixed points of Δ and let $F(f)$ denote the expression $f(b)/f(c), f \in U_p$. We prove now

Lemma 1. *The functions corresponding to the non-singular boundary points of the set E satisfy the differential equation*

(3.1) $$e^{-i\theta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \frac{(f(b)-f(c))f(\zeta)}{(f(b)-f(\zeta))(f(c)-f(\zeta))} = Q(\zeta)$$

for $\zeta \in \Delta\setminus\{p\}$; *here* $\theta \in \langle 0, 2\pi\rangle$ *and* $Q(\zeta)$ *is a rational function such that*

(3.2) $$Q(\zeta) \geqslant 0, |\zeta| = 1$$

holds.

Proof. Suppose that f corresponds to a non-singular boundary point of E. Then for a suitably chosen $a, a \in \mathscr{C}E$ we have

(3.3) $$|F(f)-a| = \min_{g\in U_p}|F(g)-a|$$

Let f^* be given by (2.1). Then

$$|F(f^*)-a|^2 = |F(f)-a|^2 - \lambda|F(f)-a|\mathscr{R}\left\{A\left[\frac{P(b,\zeta)-P(c,\zeta)}{f_b-f_c}e^{-i\theta} + \right.\right.$$
$$\left.\left. + e^{i\theta}\frac{\overline{P(b,\bar{\zeta}^{-1})-P(c,\bar{\zeta}^{-1})}}{f_b-f_c} + e^{-i\theta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2\frac{(f_b-f_c)f(\zeta)}{(f_b-f(\zeta))(f_c-f(\zeta))}\right]\right\} + O(\lambda^2)$$

where $f_b = f(b), f_c = f(c), f = f(\zeta)$ and $\theta = \arg[F(f)-a]$.

In view of (3.3) the real part of the expression in the braces must be equal to zero for each $\zeta \in \Delta\setminus\{p\}$. Since $\arg A$ can be chosen in an arbitrary manner we obtain the condition

(3.4) $$e^{-i\theta}\left(\frac{\zeta f'}{f}\right)^2\frac{(f_c-f_b)f}{(f_c-f)(f_b-f)} = e^{-i\theta}\frac{P(b,\zeta)-P(c,\zeta)}{f_b-f_c} + $$
$$+ e^{i\theta}\frac{\overline{P(b,\bar{\zeta}^{-1})-P(c,\bar{\zeta}^{-1})}}{f_b-f_c} \equiv Q(\zeta)$$

which holds for $\zeta \in \Delta\setminus\{p\}$.

Now let us apply the formula (2.2) to the extremal function f. Then

$$|F(f^{\Delta})-a|^2 = |F(f)-a|^2 + \lambda|F(f)-a|\mathscr{R}\left\{e^{-i\theta}\frac{P(b,\zeta_0)-P(c,\zeta_0)}{f_b-f_c}\right\} + O(\lambda^2)$$

from which we obtain

$$\mathscr{R}\left\{e^{-i\theta}\frac{P(b,\zeta_0)-P(c,\zeta_0)}{f_b-f_c}\right\} \geqslant 0.$$

From the last we obtain (3.2).

Now (2.3) and (3.4) yield

(3.5) $$Q(\zeta) = A_1 + A_2\frac{\zeta+b}{\zeta-b} + A_3\frac{\zeta+c}{\zeta-c} + A_4\frac{\zeta+p}{\zeta-p} +$$
$$+\bar{A}_1 + \bar{A}_2\frac{1+\bar{b}\zeta}{1-\bar{b}\zeta} + \bar{A}_3\frac{1+c\zeta}{1-c\zeta} + \bar{A}_5\frac{1+p\zeta}{1-p\zeta}.$$

Hence Lemma 1 has been proved.

If we apply (2.4) to (3.3) we can convince ourselves that the set $\mathscr{C}f(\Delta)$, where f is an extremal function, has no interior points so that the set is the whole plane slit along a finite number of arcs.

The condition (3.4) has been established for $\zeta \in \Delta \setminus \{p\}$. However, it is well-known that it holds also on $|\zeta| = 1$. Hence, the extremal functions map Δ onto the whole plane cut along a finite number of analytic arcs.

4. The form of $Q(\zeta)$

We have proved that $Q(\zeta)$ is a rational function. It is easy to see

$$\overline{Q(\bar{\zeta})} = Q(\zeta)$$

which implies that the roots of the equation $Q(\zeta) = 0$ are symmetric w.r.t. the unit circumference. Moreover (3.4) shows that $Q(\zeta) \neq 0$ for $\zeta \neq 0, \infty$, and $|\zeta| \neq 1$. Since the equation $Q(\zeta) = 0$ has at most 6 roots and the roots on $|\zeta| = 1$ have an even order of multiplicity then $Q(\zeta)$ must have the form

$$Q(\zeta) = A\frac{\zeta(\zeta-k)^2(\zeta-l)^2}{(b-\zeta)(c-\zeta)(p-\zeta)(1-p\zeta)(1-c\zeta)(1-\bar{b}\zeta)} \tag{4.11}$$

where $k = e^{i\alpha}$, $l = e^{i\beta}$ are the points on $|\zeta| = 1$ which are carried by f onto the endpoints of the arc $f(|\zeta| = 1)$. Of course, $f'(k) = f'(l) = 0$.

Hence f maps Δ onto the whole plane cut along one analytic arc with endpoints $f(k), f(l)$.

Since $f(|\zeta| = 1)$ is an analytic arc, the points k, l divide the unit circumference into two arcs a_1, a_2 with common endpoints which have the same length in the metric $|Q(e^{i\theta})|^{\frac{1}{2}}d\theta$. Hence

$$\int_{\alpha}^{\beta} |Q(e^{i\theta})|^{\frac{1}{2}}d\theta = \int_{\beta}^{\alpha+2\pi} |Q(e^{i\theta})|^{\frac{1}{2}}d\theta$$

from which we obtain

$$\int_{0}^{2\pi} \varphi(\theta)\sin\frac{\theta-\beta}{2}\sin\frac{\theta-\alpha}{2}d\theta = 0, \tag{4.2}$$

where

$$\varphi(\theta) = |(b-e^{i\theta})(c-e^{i\theta})(p-e^{i\theta})|^{-1}.$$

Now (4.2) can be written in the form

$$k+l = \bar{D}+Dk\cdot l \tag{4.3}$$

where

$$D = \int_{0}^{2\pi} e^{i\theta}\varphi(\theta)d\theta \Big/ \int_{0}^{2\pi} \varphi(\theta)d\theta. \tag{4.4}$$

On the other hand $Q(e^{i\theta}) \geq 0$, so that it follows that

$$k \cdot l = \bar{A}|A|^{-1} = e^{-i\varphi}$$

holds. Finally we find that $Q(\zeta)$ has the form

$$Q(\zeta) = |A| \frac{\overline{kl}\zeta(\zeta-k)^2(\zeta-l)^2}{(b-\zeta)(c-\zeta)(p-\zeta)(1-p\zeta)(1-\bar{c}\zeta)(1-\bar{b}\zeta)} \tag{4.5}$$

Now, if we compare the Laurent coefficients of both sides of (2.1) near the point $\zeta = 0$ and if we use (3.5), then we obtain

$$A = e^{-i\theta}\frac{f_b - f_c}{f_b \cdot f_c} pbck^{-2}l^{-2}.$$

Thus we have proved the following:

Theorem 1. *If $b, c, b \neq c \neq p \neq 0$ are given points of the unit disc and if $t \in \langle 0, 2\pi\rangle$, then the functions corresponding to the non-singular boundary points of E satisfy the differential equation*

(4.6)

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \frac{f_b \cdot f_c \cdot f(\zeta)}{(f_b - f_\zeta)(f_c - f(\zeta))} = \frac{bpc\zeta\left(1-(D+\bar{D}e^{it})\zeta + e^{+it}\zeta^2\right)^2}{(b-\zeta)(c-\zeta)(p-\zeta)(1-p\zeta)(1-\bar{c}\zeta)(1-\bar{b}\zeta)}$$

and map Δ onto the whole plane cut along one analytic arc.

5. The region E

In this section we shall determine the region of variability of the ratio $f(b)|f(c)$ within the class U_p.

Let z_0, z_1, z_2, z_3 denote the points $0, b, c, p$, respectively,

$$R(\zeta) = \left[\frac{z_1 z_2 z_3}{\zeta(z_1-\zeta)(z_2-\zeta)(z_3-\zeta)(1-\bar{z}_1\zeta)(1-\bar{z}_2\zeta)(1-\bar{z}_3\zeta)}\right]^{\frac{1}{2}}$$

(5.1)

$$S(\zeta) = 1-(D+\bar{D}e^{it})\zeta + e^{it}\zeta^2$$

and let F_k be the value of the integral $\int\limits_\gamma R(\zeta)(\zeta)d\zeta \equiv I(z_k)$ taken along a closed curve γ situated in Δ that starts from $z = l$ and incloses the single critical point z_k ($k = 0$ to 3). This closed curve can be reduced to a loop formed by the straight line lz_k, the circle of infinitesimal radius about z_k, and the straight line $z_k l$.

Then

$$F_k = 2\int\limits_l^{z_k} S(\zeta)R(\zeta)d\zeta, \quad k = 0, 1, 2, 3$$

where the integral is taken along the straight line lz_k.

Let

$$\Omega_k = F_k - F_0, \quad k = 1, 2, 3. \tag{5.2}$$

It is well-known [6] that

$$I(\gamma) = I(\zeta) + \sum_{k=1}^{3} m_k \Omega_k$$

or

$$I(\gamma) = F_3 - I(\zeta) + \sum_{k=1}^{3} m_k' \Omega_k$$

where $I(\gamma)$ denotes the integral taken along a curve γ jointing l to ζ.

In the case under considerations there is the possibility of eliminating one of the constants Ω_k.

Since (4.2) holds, we have $\int\limits_{|\zeta|=1} R(\zeta)S(\zeta)d\zeta = 0$ and hence

$$F_3 - F_2 + F_1 - F_0 = 0 = \int\limits_{|\zeta|=1} R(\zeta)S(\zeta)d\zeta$$

Therefore

$$\Omega_3 - \Omega_2 + \Omega_1 = 0$$

Thus (5.3) takes the form

$$I(\gamma) = I(\zeta) + \sum_{1}^{2} m_k \Omega_k$$

(5.4)

$$I(\zeta) = F_3 - I(\zeta) + \sum_{1}^{2} m_k \Omega_k$$

Let us write (4.1) in the form

$$\frac{v_0 dv}{\sqrt{v(v_0 - v)(v_0 - qv)}} = S(\zeta)R(\zeta)d\zeta$$

where $v_0 = f(b)$, $q = f(b)/f(c)$, $v = f(\zeta)$.

If we set

$$v = v_0\left(x + \frac{1+q}{3q}\right)$$

$$g_2 = \frac{4(q^2 - q + 1)}{3q^2}, \quad g_3 = \frac{4(1+q)(q-2)(2q-1)}{27q^3}$$

then we have

$$\frac{2dx}{\sqrt{v_0}\sqrt{4x^3 - g_2 x - g_3}} = R(\zeta)S(\zeta)d\zeta \tag{5.5}$$

and $g_2^3 - 27 g_3^2 \neq 0$.

Let $\wp(u;\Omega_1,\Omega_2)$ be the elliptic function of Weierstrass defined by the formula

$$\wp(u;\Omega_1,\Omega_2) = u^{-2}+\sum_{-\infty}^{+\infty}{}'\,[(u+n\Omega_1+m\Omega_2)^{-2}-(n\Omega_1+m\Omega_2)^{-2}].$$

If we integrate (5.5), then we obtain

$$f(\zeta) = A\wp\Big(\int_p^\zeta S(\zeta)R(\zeta)d\zeta\Big)+B \tag{5.6}$$

where A, B are constants and we have made use of the fact that $f(p)=\infty$.

The formulas (5.4) show that we can write (5.6) in the form

$$f(\zeta) = A\wp\Big(\int_0^\zeta S(u)R(u)du+\tfrac{1}{2}(\Omega_1+\Omega_2)\Big)+B$$

and hence the function f is single valued.

Finally, because $f(0)=0$ we have

$$f(\zeta) = A\Big[\wp\Big(\int_0^\zeta S(x)R(x)dx+\tfrac{1}{2}(\Omega_1+\Omega_2)\Big)-e_2\Big] \tag{5.7}$$

where $e_2=\wp\big(\frac{1}{2}(\Omega_1+\Omega_2)\big)$ and A has to be chosen so that $f'(0)=1$.

Formula (5.7) gives us the form of the extremal functions.

If we set $\zeta=b$, $\zeta=c$ then we obtain $(k=1,3)$

$$\frac{f(b)}{f(c)} = \frac{e_3-e_2}{e_1-e_2},\ e_k=\wp\Big(\frac{\Omega_k}{2}\Big) \tag{5.8}$$

Let $\lambda(\tau)$ be the modular function defined as a conformal mapping of the domain $\{0<\mathscr{R}\tau<1\}\cap\{|\tau-\frac{1}{2}|>\frac{1}{2}\}$ onto the upper half plane such that $\lambda(0)=1$, $\lambda(1)=\infty$, $\lambda(\infty)=0$.

Then [1]

$$\frac{e_2-e_3}{e_1-e_3} = \frac{\lambda(\tau)}{\lambda(\tau)-1} = \lambda(\tau+1) \tag{5.9}$$

which combines with (5.8) to yield

$$\frac{f(b)}{f(c)} = \lambda\Big(1-\frac{\Omega_1(t)}{\Omega_2(t)}\Big) = w(t) \tag{5.10}$$

As t varies from 0 to 2π the quotient $\Omega_1(t)/\Omega_2(t)$ describes a circle. Hence, the set of all bundary points of the region E lies on the curve (5.10).

Thus we have established the following

Theorem 2. *The region E is a closed set whose boundary is given by the equation*

$$w(t) = \lambda\left(1+\frac{\Omega_1(t)}{\Omega_2(t)}\right), \quad t \in \langle 0, 2\pi\rangle$$

where λ is the modular function defined by (5.9) and Ω_1, Ω_2, $\operatorname{Im}\Omega_2/\Omega_1 > 0$ are given by the formulas

$$\Omega_k = \int_{\gamma_k} S(x)R(x)dx, \quad k = 1, 2$$

γ_k is a loop that incloses the points 0 and z_k ($k = 1, 2$) and that leaves the other critical points outside.

6. The limit case $p = 1$

It follows from (4.3) that $\lim_{p=1} D(p) = 1$. Hence $k = 1$ or $l = 1$ and we obtain

$$\frac{f(b)}{f(c)} = \lambda\left(1+\frac{\Omega_1'(t)}{\Omega_2'(t)}\right); \quad f \in S$$

where

$$\Omega_k' = \int_{\gamma_k} \frac{(1-e^{it}x)\,dx}{\sqrt{x(z_1-x)(z_2-x)(1-\bar{z}_1, x)(1-\bar{z}_2 x)}}$$

This is a well-known result due to J. G. Krzyż [3].

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STRESZCZENIE

W pracy podano dokładny obszar zmienności wyrażenia $f(b)/f(c)$ w klasie meromorficznych i jednolistnych funkcji.

РЕЗЮМЕ

В работе определена точная область значений выражения $f(b)/f(c)$ в классе мероморфных и однолистных функций.

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