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On a Family of Starlike Functions

O pewnej rodzinie funkcji gwiaździstych

О некотором классе звездообразных функций

1. Introduction

Let S denote the class of functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

analytic and univalent in the disc K_1; here K_r denotes the disc $\{z: |z| < r\}$. Let S^* be the subclass of S consisting of functions mapping the disc K_1 onto domains starlike with respect to the origin. It is well known that $f \in S^*$, iff f has the form (1.1) and satisfies the condition

$$\operatorname{re} \frac{zf'(z)}{f(z)} > 0, \quad \text{for} \quad z \in K_1, \tag{1.2}$$

or the equivalent condition

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}, \quad \text{for} \quad z \in K_1. \tag{1.3}$$

Various authors investigated the families of a-starlike functions that is functions $f \in S$, which are subject to the condition

$$\operatorname{re} \frac{zf'(z)}{f(z)} > a, \tag{1.4}$$

for $z \in K_1$ and fixed a, $0 \leqslant a < 1$, which is more restrictive than the condition (1.2). Taking $a = 0$ in (1.4) we obtain the class S^*.

It is possible to restrict the condition (1.3) in an analogous way:

Definition 1.1. A function $f \in S_a$, if it has the form (1.1) and is subject to the condition

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < a\frac{\pi}{2}, \tag{1.5}$$

for $z \in K_1$ and a fixed a, $0 < a \leqslant 1$.

A function f of the class S_a is said to be strongly starlike of order a (cf. [1]).

The aim of the present paper is to investigate the class S_a. We shall give the geometrical interpretation of functions of this family and prove a theorem connected with the circular symmetrization of strongly starlike domains. We shall use latter theorem to solve some extremal problems within the family S_a.

Let us observe that, if $0 < a_1 \leqslant a_2 \leqslant 1$, then

$$S_{a_1} \subset S_{a_2} \subset S_1 = S^*.$$

2. Strongly starlike domains and their connection with the class S_a.

Let a be a fixed number from the interval $\langle 0, 1\rangle$.

Definition 2.1. A domain D containing the origin is said to be strongly starlike of order a, if any point w_0 of the complementary set $E \setminus D$ of D is the vertex of an angle of measure $(1-a)\pi/2$ also contained in $E\setminus D$ and bisected by the radius vector through w_0.

We shall denote by G_a the family of all domains strongly starlike of order a. Let us observe that, if $0 < a_1 \leqslant a_2 \leqslant 1$, then

$$G_{a_1} \subset G_{a_2} \subset G_1,$$

where G_1 is the family of all domains starlike with respect to the origin.

We shall give now some properties of the domains of the family G_a, $a < 1$.

Theorem 2.1. *If $D \in G_a$ for $0 \leqslant a < 1$, and $w_0 \in E \setminus D$, then*

$$D \subset H(w_0, a), \tag{2.1}$$

where $H(w_0, a)$ is the Jordan domain bounded by arcs of logarithmic spirals joining the points w_0, $-w_0\exp\{\pi\tan(a\pi/2)\}$ and intersecting the radius vectors at an angle $(1-a)\pi/2$.

Corollary 1.1. *If $D \in G_a$ for some fixed a, $0 \leqslant a < 1$, then*

$$D \subset K_\varrho, \tag{2.2}$$

where

$$\varrho = w_0\exp\{\pi\tan(a\pi/2)\},$$

and w_0 is an arbitrary point belonging to $E \setminus D$.

Theorem 2.2. *If $\{D_n\}$ is an increasing sequence of domains of the family G_a which tends in Carathéodory's sense to a domain D, then also D belongs to the family G_a.*

The connection between the class S_a and the family G_a is illustrated by the following theorems:

Theorem 2.3. *If $f \in S_a$, then for each r, $0 < r \leqslant 1$, the domains $F(K_r)$ belong to the family G_a.*

Theorem 2.4. *If a function $g(z) = a_1 z + a_2 z^2 + \dots$ is analytic and univalent in K_1 and maps K_1 onto a domain D of the family G_a, then the function $f(z) = g(z)/a_1$ belongs to the class S_a.*

These theorems were proved by using the connection between strongly starlike domains of the order a and the so-called β-spirallike domains.

Theorem 2.5. *If $f \in S_a$, then the domains $f(K_r)$, $0 < r \leqslant 1$, have the following property: each logarithmic spiral with the focus at the origin intersecting the radius vectors at an angle not less than $(1-a)\pi/2$, consists of two arcs one of which lies entirely inside $f(K_r)$, whereas the other one lies entirely outside $f(K_r)$.*

In other words: domains of the family G_a are β-spirallike domains for each β, $|\beta| \leqslant (1-a)\pi/2$. From this we obtain

Corollary 2.2. *The class S_a is an intersection of two classes:*

$$S_a = \check{S}_{(1-a)\pi/2} \cap \check{S}_{-(1-a)\pi/2}, \tag{2.3}$$

where $\check{S}_\beta$ is a subclass of β-spirallike functions in Špaček's sense [5], i.e. $f \in S$ and f satisfies the condition:

$$\operatorname{re}\left\{e^{-i\beta}\frac{zf'(z)}{f(z)}\right\} > 0.$$

3. Circular symmetrization of domains of the family G_a

Let D be a domain containing the origin and let D^* be a domain obtained from D by circular symmetrization (for the definition of circular symmetrization cf. e.g. [2]).

Z. Lewandowski showed [4] that $D \in G_1$ implies $D^* \in G_1$. As shown by the present author in his thesis, an analogous result also holds for strongly starlike domains of order a.

Theorem 3.1. *If $D \in G_a$ then also D^* belongs to G_a.*

Theorem 3.1. leads to the solution of an extremal problem similar to that solved by J. A. Jenkins [3] for the whole class S.

Let $L(r, f)$, $f \in S$, denote the linear measure of the circle $|w| = r$ omitted by the values of the function $w = f(z)$, $z \in K_1$. J. A. Jenkins [3] determined the exact value

$$l(r) = \sup_{f \in S} L(r, f).$$

Z. Lewandowski [4] solved an analogous result for the class $S^* = S_1$. We shall give here an analogous result for the class S_a. To this end we first state:

Theorem 3.2. *If $f \in S_a$, $0 < a < 1$, then the domain $F(K_1)$ contains the disc K_r, where*

$$r_a = \exp\int_0^1 \left\{\left[\frac{1-t}{1+t}\right]^a - 1\right\}\frac{dt}{t} = \exp\left\{\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} - \frac{\Gamma'\left(\frac{1+a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)}\right\}$$

and is contained in the disc K_ϱ, where $\varrho = \exp\{\pi\tan(a\pi/2)\}$.

Theorem 3.3. *For each function $f \in S_a$ and $r \in \langle r_a, 1\rangle$ the inequality*

$$L(r, f) \leqslant 2r\varphi(r) \tag{3.2}$$

holds, where $\varphi(r)$ is determined by the system of equations

$$\ln r = \int_0^1 \left\{\left[\frac{\sqrt{1-2t\cos\gamma + t^2}}{1-t}\right]^a - 1\right\}\frac{dt}{t},$$

$$\varphi(r) = \int_0^\gamma \left[\frac{\cos\theta - \cos\gamma}{\cos(\theta/2)}\right]^a d\theta.$$

The equality in (3.2) is obtained for the function

$$F(z) = z\exp\int_0^z \left\{\left[\frac{\sqrt{1-2u\cos + u^2}}{1+u}\right]^a - 1\right\}\frac{du}{u},$$

which maps K_1 onto the domain $F(K_1)$ bounded by a suitable arc of the circle $|w| = r$ of length $2r\varphi(r)$ and by two arcs of logarithmic spirals emanating from the end points of this circular arc respectively and ending at their point of intersection.

4. A connection between the class S_a and the class P of functions of positive real part

Let P denote the class of functions $p(z)$ of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{4.1}$$

that are analytic in the unit disc K_1 and satisfy the condition

$$\operatorname{re} p(z) > 0, \quad \text{for} \quad z \in K_1. \tag{4.2}$$

It follows from the definition of the class S_a that if $f \in S_a$ then $p(z) = [zf'(z)/f(z)]^{1/a}$ belongs to P and conversely, if $f(z) = z + a_2 z^2 + \ldots$ satisfies the condition $zf'(z)/f(z) = p^a(z)$, where $p \in P$ then f belongs to S_a.

Copsequently we obtain

Theorem 4.1. *A function f belongs to the class S_a if and only if there exists a function $p \in P$ such that*

$$f(z) = z\exp\int_0^z \frac{p^a(u)-1}{u}\,du, \tag{4.3}$$

holds.

Using (4.3) one can easily prove the following

Theorem 4.2. *If a function f belongs to S_a then for $|z| = r < 1$ we have*

$$\left(\frac{1-r}{1+r}\right)^a \leqslant \operatorname{re}\frac{zf'(z)}{f(z)} \leqslant \left(\frac{1+r}{1-r}\right)^a, \tag{4.4}$$

$$\left(\frac{1-r}{1+r}\right)^a \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \left(\frac{1+r}{1-r}\right)^a, \tag{4.5}$$

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant 2a\arctan r, \tag{4.6}$$

$$r\exp\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\binom{a}{k}\binom{-a}{n-k}\right]\frac{r^n}{n} \leqslant |f(z)| \leqslant r\exp\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\binom{-a}{k}\binom{a}{n-k}\right]\frac{r^n}{n}, \tag{4.7}$$

$$\left(\frac{1-r}{1+r}\right)^a\exp\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\binom{a}{k}\binom{-a}{n-k}\right]\frac{r^n}{n} \leqslant |f'(z)| \leqslant \left(\frac{1-r}{1+r}\right)^a\exp\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\binom{-a}{k}\binom{a}{n-k}\right]\frac{r^n}{n}. \tag{4.8}$$

The estimates (4.4) – (4.8) *are sharp. The extremal functions have the form*

$$F(z) = z\exp\int_0^z\left[\left(\frac{1+\varepsilon u}{1-\varepsilon u}\right)^a - 1\right]\frac{du}{u} \tag{4.9}$$

where $|\varepsilon| = 1$.

5. Some relations between S, S^*_β, S^*, S_a, S^c

Let S^c denote the class of functions of the form (1.1) subject to the condition

$$\operatorname{re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for} \quad z \in K_1. \tag{5.1}$$

A function f is is said to be starlike of order a, strongly starlike of order a and convex in the disc K_R if it satisfies there the conditions (1.4), (1.5) and (5.1), respectively.

The radii of starlikeness of order a, strong starlikeness of order a and the radii of convexity within the classes S, S^*_β, S^*, S_a, and S^c are given by the follwing theorems.

Theorem 5.1. *If $f \in S$ then f is strongly starlike of order a at least in the disc $K_{R(a)}$ where*

$$R(a) = \operatorname{th}\frac{a\pi}{4}. \tag{5.2}$$

Theorem 5.2. *If $f \in S^*_\beta$ then f is strongly starlike of order a at least in the disc $K_{R^*(\beta,a)}$, where*

$$R^*(\beta, a) = \begin{cases} 1-\beta-\sqrt{(1-\beta)^2-(1-2\beta)\sin^2 a\pi}, & \text{for } \beta \neq \frac{1}{2} \\ \sin\frac{a\pi}{2} & \text{for } \beta = \frac{1}{2} \end{cases} \tag{5.3}$$

The number $R^*(\beta, a)$ is the best possible one. The extremal functions have the form

$$F_\beta(z) = z(1-\varepsilon z)^{-2(1-\beta)} \quad \text{where} \quad |\varepsilon| = 1. \tag{5.4}$$

Theorem 5.3. *If $f \in S_a$ then f is starlike of order β at least in the disc $K_{R_*(a,\beta)}$ where $R_*(a, \beta) = (1-\beta^{1/a})/(1+\beta^{1/a})$. The number $R_*(a, \beta)$ is the best possible one and the extremal functions have the form* (4.9).

Theorem 5.4. *Each function f of the class S_γ is strongly starlike of order a at least in the disc $K_{R(\gamma,a)}$, where*

$$R(\gamma, a) = \begin{cases} \tan\frac{a\pi}{4\gamma} & \text{for } a < \gamma, \\ 1 & \text{for } a \geqslant \gamma. \end{cases}$$

The radius $R(\gamma, a)$ is the best possible one and the extremal functions have the form (4.9).

Theorem 5.5. *Each function f of the class S_a is convex at least in the disc $K_{R^c(a)}$ where $R^c(a)$ is the smallest positive root of the equation*

$$(1-r)^{1+a}(1+r)^{1-a}-2ar = 0.$$

The radius $R^c(a)$ is the best possible one and the extremal functions have the form (4.9).

Taking $\beta = 1/2$ in Theorem 5.9. in view of the relation $S^c \subset S_{1/2}$ we obtain:

Theorem 5.6. *If $f \in S^c$ then f is strongly starlike of order a at least in the disc $K_{R_c(a)}$, where*

$$R_c(a) = \sin \frac{a\pi}{2}$$

and the value $R_c(a)$ is the best possible.

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STRESZCZENIE

W pracy tej rozpatrywana jest pewna podklasa funkcji gwiaździstych określona warunkiem (1.5). Podana jest interpretacja geometryczna tej klasy oraz twierdzenia dotyczące symetryzacji kołowej i promieni wypukłości, gwiaździstości, a-gwiaździstości itp.

РЕЗЮМЕ

В работе рассмотрен некоторый подкласс звездообразных функций, определенных условием (1.5). Дана геометрическая интерпретация этого класса и теоремы о симметризации Пойа и радиусах выпуклостей, зведообразности, a-зведообразности и т. п.