ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXII/XXIII/XXIV, 21 SECTIO A 1968/1969/1970

Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA
Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

MAXWELL O. READE and ELIGIUSZ J. ZŁOTKIEWICZ

On the Equation $f(z) = pf(a)$ in Certain Classes of Analytic Functions

O równaniu $f(z) = pf(a)$ dla pewnych klas funkcji analitycznych

Уравнение $f(z) = pf(a)$ для некоторых классов аналитических функций

Let $\mathscr{K}$ denote a fixed and compact class of functions analytic in the unit disc Δ and let p and a be fixed complex numbers $p \neq 0, 1, 0 < |a| < 1$. It is clear that for p, a and f fixed the equation

$$(1) \qquad f(z) = pf(a)$$

may have a solution z_f in the disc Δ. This leads to the following extremal problem. For given p and $a, p \neq 0, 1\ 0 < |a| < 1$ determine

$$m(p, a, \mathscr{K}) = \inf_{f \in \mathscr{K}} \{w: w = |z_f|\}$$

Not so long ago, P. Mocanu [2] considered this problem for a class S of functions analytic and univalent in Δ. More recently J. Kaczmarski [1] found $m(p, a, \mathscr{K})$ for the class S_R of typically-real univalent functions under the additional assumption that p is real.

In the present note we shall give more simple proofs of the results due to J. Kaczmarski and we shall solve (1) for the classes of functions that are

(i) starlike of order a in the unit disc Δ,
(ii) typically real in Δ,
(iii) spiral-like in Δ.

If a, z and $\mathscr{K}$ are given then the set $D(a, z, \mathscr{K}) = \{w: w = f(z)/f(a) \wedge f \in \mathscr{K}\}$ is called the region of variability of the ratio $f(z)/f(a)$ within the class $\mathscr{K}$.

In terms of the region of variability, the problem of determining of $m(p, a, \mathscr{K})$ can be considered in the following way.

If p, a are given then (1) has a solution in $\mathscr{K}$ if and only if $p \in D(a, p, \mathscr{K})$ holds for some $z \in \Delta$.

This is the key to our method and leads to our solutions.

Let $S_a^*, S^*, S^c, \check{S}$, be the classes of functions $f(z) = z + a_2 z^2 + \ldots$ that are univalent and starlike of order a, starlike, convex or spirallike in the unit disc, respectively.

Let TR denote the class of typically-real functions in Δ.

If we make use of the structural formulas for the classes S_a^*, $\check{S}$, and TR we easily obtain the corresponding sets $D(a, p)$ and we establish the following theorems.

Theorem 1. *If a, p are complex constants, $p \neq 0, 1$, $0 < |a| < 1$ and if $\beta = [2(1-a)]^{1}$ then*

$$m(a, p, S_a^*) = \inf\left\{|z| : \left|z - a\left(\frac{z}{ap}\right)^{\beta}\right| = \left|1 - \left(\frac{z}{ap}\right)^{\beta}\right|\right\}$$

The extremal functions have the form

$$(1-e^{i\theta}z)^{-2(1-a)}, 0 \leqslant a < 1, \theta - \text{real}.$$

Taking $a = 0$ or $a = 1/2$ after some elementary calculations we obtain the following theorems

Theorem 2. $$m(a, p, S_0^*) = m(a, p, S^*) = \frac{1+|a|^2 - 2|a(p-1)|}{2|ap|} - \sqrt{\left[\frac{1+|a|^2-2|a(p-1)|}{2|ap|}\right]^2 - 1}.$$

Theorem 3. $m(a, p, S_{1/2}^*) = m(a, p, S^c) = \dfrac{|ap|}{1+|a(p-1)|}$. *The extremal function has the form* $z(1-xz)^{-1}$, $x = -\dfrac{|a(p-1)|}{a(p-1)}$.

Let $pf(\Delta)$ denote an image-domain of $f(\Delta)$ under the transformation $W = pw$.

If we make use of Th. 2 and Th. 3 we establish

Theorem 2′. *For each function f of the class S^* we have*

$$f(|z| < r^*) \subset pf(\Delta)$$

where

$$r^* = \frac{1+|p-1|}{|p|} - \sqrt{\frac{(1+|p-1|)^2}{|p|^2} - 1}$$

The number r^* is the best possible one.

Theorem 3′. *For each function f of the class S^c we have*

$$f(|z| < r^c) \subset pf(\Delta)$$

where

$$r^c = |p|(1+|p-1|)^{-1}$$

The value r^c is exact one.

Theorem 4. *Suppose that a, p, γ, s are fixed complex numbers $p \neq 0,1$, $0 < |a| < 1$, $|\gamma| < \pi/2$, $s = -e^{i\gamma}/\cos\gamma$. Then*

$$m(a, p, \check{S}) = \inf\left\{|z|: \left|z - a\left(\frac{z}{ap}\right)^s\right| = \left|1 - \left(\frac{z}{ap}\right)^s\right|\right\}.$$

Theorem 5. *If a and p are fixed real constants then*

$$m(a, p, TR) = \begin{cases} \left|-1 + (1+a)\dfrac{1+a-\sqrt{(1+a)^2-4ap}}{2ap}\right|, & a(p-1) < 0 \\ \left|1 + (1-a)\dfrac{1-a-\sqrt{(1-a)^2+4ap}}{2ap}\right|, & a(p-1|) > 0 \end{cases}$$

and the bounds are attained by the Koebe functions $z(1 \pm z)^{-2}$, resp.

The details and some further results will be published in Mathematica, Cluj, (1971).

REFERENCES

[1] Kaczmarski, J., *Sur l'equation $f(z) = pf(c)$ dans la classe des fonctions univalentes a coefficientes reels*, Bull. Acad. Pol. Sci. 15 (1967), 245–251.

[2] Mocanu, P., *On the equation $f(z) = af(a)$ in the class of univalent functions*, Mathematica, Cluj, 6 (1964), 63–79.

STRESZCZENIE

W doniesieniu podaje się bez dowodów rozwiązania problemu wyznaczania kresu dolnego rozwiązań równania $f(z) = pf(a)$ w pewnych rodzinach funkcji analitycznych w kole jednostkowym.

РЕЗЮМЕ

В работе приводятся без доказательства решения проблемы определения точной нижней границы решений уравнения $f(z) = pf(a)$ в некоторых классах аналитических функций.