ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXII/XXIII/XXIV, 20 SECTIO A 1968/1969/1970

Instytut Matematyki, Uniwersytet Jagielloński, Kraków

WIESŁAW PLEŚNIAK

Quasianalytic Functions of Several Variables

Funkcje quasianalityczne wielu zmiennych

Квазианалитические функции многих переменных

Let E be a compact set in the space C^n of n complex variables and let $\mathscr{C}(E)$ denote the Banach space of complex functions continuous in E with the norm $\|f\|_E = \max_{z\in E}|f(z)|$ for $f\in\mathscr{C}(E)$. Let us denote by $\mathscr{E}_\nu(f, E)$ the ν-th measure of the Čebyšev best approximation to $f\in\mathscr{C}(E)$ on E by polynomials of n complex variables $z = (z_1, \ldots, z_n)$, i.e.

$$\mathscr{E}_\nu(f, E) = \inf_{P_\nu}\|f-P_\nu\|_E$$

where inf is taken over all the polynomials P_ν of degree $\leqslant \nu$.

Definition 1. We say that a function $f\in\mathscr{C}(E)$ is quasianalytic on E in Bernstein's sense (and write $f\in\mathscr{B}(E)$) if

$$\liminf_{\nu\to\infty}\sqrt[\nu]{\mathscr{E}_\nu(f, E)} < 1.$$

The term "quasianalytic" arises from the following identity principle proved by Bernstein [1]:

If E and I are compact intervals in the space R of real numbers and if $I\subset E$, then for every function $f\in\mathscr{B}(E)$ we have

$$f = 0 \text{ in } I \Rightarrow f = 0 \text{ in } E.$$

Szmuszkowiczówna [5] proved that the interval I in the above result could be replaced by any compact subset of E with the positive transfinite diameter. It appears that the identity principle can be extended on quasianalytic functions of several complex variables. In order to give this extension let us denote by Φ the extremal function of a compact set E in C^n introduced by Siciak [4]:

$$\Phi(z; E) = \sup_{\nu\geqslant 1}\{\sup\{|P_\nu(z)|^{1/\nu}: P_\nu \text{ is a polynomial in } z = (z_1, \ldots, z_n)$$

such that $\deg P_\nu \leqslant \nu$ and $\|P_\nu\|_E \leqslant 1\}\}$, $z \in C^n$.

The following theorem holds true [2]

Theorem 1. *Let a continuum E in C^n be a sum $E = E_1 \cup \ldots \cup E_m$, where $E_j = E_1^j \times \ldots \times E_n^j$ for $j = 1, \ldots, m$, E_k^j $(k = 1, \ldots, n)$ being continua not reduced to a point in the complex z_k-plane, respectively. Let I be a compact subset of E such that the function $\Phi(z; I)$ is continuous at a point $\mathring{z} \in I$. Then every function $f \in \mathscr{B}(E)$ vanishing on I is identically equal to zero.*

One can easily see that the proposition of Theorem 1 holds true if we replace E by the closure of a bounded domain in R^n (treated as a subset of C^n). It is known that for every compact set I in C with the positive transfinite diameter the extremal function $\Phi(z; I)$ is continuous at a point $\mathring{z} \in I$. Hence Theorem 1, generalizes the result of Szmuszkowiczówna. A more general statement of Theorem 1 is given in [2].

Let $\{\nu_k\}$ be a fixed increasing sequence of positive integers. Let us denote by $\mathscr{B}(E, [\{\nu_k\}])$ a set of functions $f \in \mathscr{B}(E)$ such that

$$\lim_{k\to\infty} \sqrt[\mu_k]{\mathscr{E}_{\mu_k}(f, B)} < 1$$

for an increasing sequence $\{\mu_k\}$ such that $1/M < \mu_k/\nu_k < M$ for $k = 1, 2, \ldots$, M being a positive constant independent of k. The set $\mathscr{B}(E, [\{\nu_k\}])$ is a ring. If E satisfies conditions of Theorem, 1 then the ring $\mathscr{B}(E, [\{\nu_k\}])$ is a domain of integrity.

Definition 2. We say that a function f continuous in an open set G in R^n is locally quasianalytic in G if for every point $x \in G$ there exists an n-dimensional interval E_x such that $x \in E_x \subset G$ and $f \in \mathscr{B}(E_x, [\{\nu_k\}])$.

One can prove [3] the following

Theorem 2. *A function f is locally quasianalytic in an open set G in R^n if and only if $f \in \mathscr{B}(E, [\{\nu_k\}])$ for every compact set E in G.*

Theorem 2 generalizes the following result of Bernstein [1]:

If E and F are compact intervals in R such that $E \cap \operatorname{int} F \neq \emptyset$ and if $f \in \mathscr{B}(E, [\{\nu_k\}])$ and $f \in \mathscr{B}(F, [\{\nu_k\}])$, then $f \in \mathscr{B}(E \cup F, [\{\nu_k\}])$.

REFERENCES

[1] Бернштейн С. К., *Собрание сочинений*, Издат. АН СССР 1 (1952).

[2] Pleśniak, W., *Quasianalytic functions of several complex variables*, Zeszyty Naukowe UJ 15(1971), 135-145.

[3] —, *Locally quasianalytic functions in R^n*, (to appear).

[4] Siciak, J., *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. 105 (2), (1962), 322–357.

[5] Szmuszkowiczówna, H., *Un théorème sur les polynômes et son application à la théorie des fonctions quasianalytiques*, C. R. Acad. Sci. Paris 198 (1934), 1119–1120.

STRESZCZENIE

Autor rozważa funkcje quasianalityczne w C^n w sensie Bernsteina (Definicja 1) i otrzymuje dla nich twierdzenie o identyczności (Twierdzenie 1).

РЕЗЮМЕ

Автор рассматривает квазианалитические функции в C^n в смысле Бернштейна (Определение 1) и получает теорему о тождестве (Теорема 1).