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On the Boundary Correspondence under Quasiconformal Mappings in Space

O odpowiedniości punktów brzegowych przy odwzorowaniach quasikonforemnych w przestrzeni

Соответствие границ для квазиконформных отображений в пространстве

In his paper [1], as the corollary of a more general theorem, F. W. Gehring stated the following result (see [1], p. 21):

If $y(x)$ is a quasiconformal mapping of a sphere D and if $y(x)$ converges to P' as x converges to $P \in \partial D$ along some endcut γ of D, then $y(x)$ converges to P' as x converges to P in a cone.

Using results and methods of F. W. Gehring we prove that the existence of the limit in a cone follows from the existence of the limit on a sufficiently "dense" sequence of points lying in a cone. By an example we also prove that the obtained density for the sequence of mentioned points is the best possible.

The notations are in accordance with those used in [1].

1. Let K_φ be a cone with angle 2φ, vertex at the origin $0 = P$ and axis x_3 in the Euclidean space E^3. We say that a sequence of points x_n is "q-dense" if the sequence of norms $|x_n| = a_n$ is of the type of geometric progression q^n, namely, such that $\lim(a_{n+1}/a_n) = q$, $0 < q < 1$. Further we shall speak about the sequences with norms $|x_n| = q^n$ but, of course, this is not restriction which is essential.

It is easy to prove following two statements:

Lemma 1. *Let $0 < q_0 < 1$, $a > 0$, let π be the plane $x_3 = 0$ and $D = \{x;\, q_0^2 a < x < a\}$. Then, for arbitrary cone K_φ there exists a constant $c(\varphi, q_0)$ such that for every pair of points $P, Q \in K_\varphi \cap D$, lying on one line through the origin we have that*

$$\frac{|P-Q|}{\varrho(P,\pi)} \leqslant c < 1.$$

Lemma 2. *Assertion that, for one quasiconformal mapping, from the existence of the limit on some sequence $x_n \in K_{\varphi_1}$ follows the existence of the limit in the cone K_{φ_1} is equivalent with the assertion that, for an other quasiconformal mapping, from the existence of the limit on a sequence of points $x'_n \in K_{\varphi_2}$ follows the existence of the limit in the cone K_{φ_2}, where* $|x_n| = |x'_n|$, $0 < \varphi_1, \varphi_2 < \frac{\pi}{2}$.

Using Lemma 9 from [1] and the Theorem 11 from [2], with our two Lemmas we obtain the result:

Theorem. *Let $y(x)$ be a quasiconformal mapping of the half-space $x_3 > 0$, $0 < \varphi < \frac{\pi}{2}$ and let $x_n \in K_\varphi$, $x_n \to 0$ with $n \to \infty$ be a sequence of points such that $|x_n| = q^n$. Suppose that there exists the limit $\lim_{n\to\infty} y(x_n) = A$. Then $y(x) \to A$ as $x \to 0$ in the cone.*

As the angle φ is arbitrary, with $0 < \varphi < \frac{\pi}{2}$, we have that, from the existence of the limit on a nontangential "q-dense" sequence follows the existence in every cone.

The obtained result for the "density" of points x_n is the *best possible*. We shall prove this by an example. We start with the example in the plane and then construct the example in the space.

Example. Given in the ζ — plane, $\zeta = (\xi, \eta)$ the domain

$$G = \{(\xi, \eta): -\xi^2 < \eta < -\xi^2(1-\xi^\varepsilon), 0 < \xi, \varepsilon > 0\}$$

We map it with the function

$$u + iv = w = e^{i\left(\frac{1}{\zeta} - i\right)}$$

onto the domain Δ in the W-plane which representes the domain between two spirals around the unit circle. Let w_n be the sequence of points lying on the $v = 0$ axis and converge to the point $w = 1$, such that in every coil lies one of them. These points are the images of points ζ_n which converge to the origin and whose real parts, for sufficiently large n, are of the same order as $\frac{1}{2n\pi}$, $\xi_n \approx \frac{1}{2n\pi}$. Here, as in further, we use the symbol to characterize the fact that two quantities are of the same order.

On the other side, we can map a subdomain of G quasiconformally onto a semispherical neighbourhood of the origin in the z-plane:

$$D = \{(x, y); x^2 + y^2 < R^2, y > 0\}.$$

Now we are going to find the approximate value for the modulus of a subdomain of G. It can be find as follows.

Let n_0 be a sufficiently large positive integer and let T_0 be the point on the η-axis, such that, if T is the point in which the tangent on the curve $\eta = -\xi^2$, which passes through the point T_0, touches it, than we have that $T_0T = T_0\zeta_{n_0}$. Let T_1 be the point on the η-axis, between T_0 and the origin. The segment T_1T cuts the curve $\eta = -\xi^2$ in the point P_1. Let γ_1 and γ_2 be two circles with center in the point T_1 and radii T_1T and T_1P_1 respectively. The circle γ_1 cuts the curve $\eta = -\xi^2(1-\xi^\varepsilon)$ in the point S_1 and the circle γ_2 cuts the segment T_1S_1 in the point Q_1. Repeating the described procedure, but starting with the points P_1 and T_2, we find the points P_2, Q_2 and S_2, e.t.c. The obtained sequence of curvilinear quadrilaterals, in fact, a sequence of ring segments, aproximates our domain G_{n_0} and, naturally, by the standard process, we can use it to find the asympthotic behaviour of the modulus of a family of curves in the domain G_{n_0}.

Let V_i be the i-th ring segment with vertices P_{i-1}, P_i, Q_i, S_i. Consider the family of curves which connect the edges $P_{i-1}P_i$ and Q_iS_i, σ_i. Denote the angle $P_{i-1}T_iS_i$ by a_i. It can be proved that the modulus of the mentioned family of curves has the value

$$\operatorname{mod}\sigma_i = \frac{1}{a_i}\ln\frac{T_iP_{i-1}}{T_iP_i}$$

As the families σ_i are disjoint, the modulus of their union is equal to the sum of moduli, i.e.

$$\operatorname{mod}\bigcup_{i=1}^{k}\sigma_i = \sum_{i=1}^{k}\frac{1}{a_i}\ln\frac{T_iP_{i-1}}{T_iP_i}$$

Taking into account that for ζ_{n_0} sufficiently close to the origin we can use the following relations, letting $k\to\infty$, if we have $P_i(\xi_i, \eta_i)$, $a_i \approx \sin a_i \approx \xi_i^{1+\varepsilon}$, $\ln(T_iP_{i-1})/(T_iP_i) \approx (\xi_{i-1}-\xi_i)/\xi_i \approx \Delta\xi_i/\xi_i$ we obtain that the modulus of the family of curves $\sigma = \lim_{k\to\infty}\bigcup_{i=1}^{k}\sigma_i$, $\max T_iT_{i-1}\to 0$ is

$$\operatorname{mod}\sigma \approx -\int_{\xi_{n_0}}^{0}\frac{d\xi}{\xi^{2+\varepsilon}} = \int_{0}^{\xi_{n_0}}\frac{d\xi}{\xi^{2+\varepsilon}}.$$

Denoting by $G_{n_0}^n$ and $(\sigma)_{n_0}^n$ the domain obtained from G_{n_0} subtracting its part contained in that of the circles γ which passes through the point ζ_n and the corresponding family of curves, we find

$$\operatorname{mod}(\sigma)_{n_0}^n \approx -\int_{\xi_{n_0}}^{\xi_n}\frac{d\xi}{\xi^{2+\varepsilon}} = \int_{\xi_n}^{\xi_{n_0}}\frac{d\xi}{\xi^{2+\varepsilon}} = \frac{1}{1+\varepsilon}\left[(\xi_n)^{-(1+\varepsilon)} - (\xi_{n_0})^{-(1+\varepsilon)}\right].$$

Under the quasiconformal mapping of a subdomain of G onto the mentioned semicircular neighbourhood of the origin in the z-plane let the domain $G^n_{n_0}$ is mapped onto the semiring $r_n < |z| < r_{n_0} \equiv y > 0$. The modulus of the family of curves $(\beta)^n_{n_0}$, the images of $(\sigma)^n_{n_0}$, is

$$\operatorname{mod}(\beta)^n_{n_0} = \frac{1}{\pi} \ln \frac{r_{n_0}}{r_n}.$$

As the modulus of the family of curves is quasiinvariant under the quasiconformal mapping we obtain that

$$r \approx r_0 (e^b)^{\xi_n^{-(1+\varepsilon)}}, \qquad b < 0,$$

and finally, as the image of ξ_n, the joint z_n lies on the circle $|z| = r_n$, and $\xi_n \approx \dfrac{1}{2n\pi}$, we obtain

$$|z_n| \approx r_0 p^{n^{(1+\varepsilon)}}, \qquad 0 < p < 1.$$

So, the sequence of points $\{\zeta_n\}$ is mapped onto a sequence of points $\{z_n\}$ such that the norm of z_n is of order $p^{n^{(1+\varepsilon)}}$ where ε is an arbitrary positive number.

Consider now the mapping which represents the composition of the inverse of mentioned quasiconformal mapping and the mentioned conformal mapping. It is quasiconformal and the limit on the sequence z_n exists, but, evidently, can not speak about the existence of the limit in a cone. The arbitrarity of ε proofs our assertion that the obtained "density" is the best possible.

Now we are going to construct the example in the space. With the domain $G^n_{n_0}$ we associate the space domain which represents a circular horn, such that its plane of symmetry is our ζ-plane and the intersection of the ζ-plane and the horn is our domain $G^n_{n_0}$. We map it quasiconformally on the space domain associated with our domain Δ_{n_0} which represents a space spiral whose plane of symmetry is our w-plane and is obtained so that the mapping which was realised, is repeated in every direction on every level of the horn G_h. On the other side we map our horn quasiconformally on a 3-semisphere, so that in one its big circle we obtain our original plane mapping and in every other direction the mapping is repeated, again on every level of the horn. So, composing two quasiconformal mappings, we obtain a quasiconformal mapping of the semisphere onto the space spiral, such that in the planes of symmetry the mapping coincides with already considered plane mappings. Thus, we have a sequence of points with norms $r_0 p^{n^{(1+\varepsilon)}}$ on which there exists the limit, but about the limit in a cone we can not speak. This proves that the obtained bound for density of points is the best possible.

REFERENCES

[1] Gehring, F. W., *The Caratheodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Scient. Fennicae, Ser. A. I. 336 (11), 1963.
[2] —, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc., 101 (1962), 353–393.

STRESZCZENIE

Opierając się na wynikach F. W. Gehringa o odpowiedniości punktów brzegowych przy odwzorowaniach quasi-konforemnych w przestrzeni autor dowodzi, że dla istnienia granicy przy zbliżaniu się wewnątrz stożka wystarczy, by istniała granica dla ciągu punktów wewnątrz stożka których normy tworzą ciąg podobny do postępu geometrycznego. Autor wykazuje na przykładzie, że otrzymane ograniczenie na „gęstość" punktów jest możliwie najlepsze.

РЕЗЮМЕ

Опираясь на результаты Ф. В. Геринга о соответствии границ для квазиконформных отображений в пространстве, автор доказал, что для существования предела при стремлении внутрь конуса достаточно, чтобы существовал предел для некоторых специальных последовательностей точек.

Автор показал на примере, что полученные условия „плотности" точек можно считать наилучшими.