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Power Series in z and $\bar{z}$

Szeregi potęgowe względem z i $\bar{z}$

Степенные ряды относительно z и $\bar{z}$

In studying non-holomorphic complex-valued functions usually a complex differential-equation is considered. Its solutions show properties which are very similar to the properties of holomorphic functions. Let us mention here the pseudo-analytic functions, introduced by L. Bers and the generalized analytic functions, introduced by I. N. Vekua.

But if we start from power series in order to study nonholomorphic functions without considering any regard to differential-equations we are led to series of the form

$$P(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm} z^n \bar{z}^m$$

We meet here phenomena which are peculiar for one-dimensional holomorphic theory but also some differences arise.

The problems considered here may be treated in two different ways. First of all some well known methods of one-dimensional holomorphic theory can be transferred to these series, and on the other hand these series may be regarded as a holomorphic function of two complex variables z_1, z_2 in the plane $z_1 = \bar{z}_2$.

We have $|z_i| = \sqrt{x_i^2+y_i^2}$, and this is the distance between the point z_i and the origin. But $|z|$ is the euclidean distance between the point $(z_1, \bar{z}_2)$ and the origin. So we have $|z| = \sqrt{x_1^2+y_1^2+x_1^2+y_1^2} = \sqrt{2}\,|z_1|$. Hence we put $z = x+iy$ with $x = \sqrt{2}x_1$, $y = \sqrt{2}y_1$. So we have the correspondence

$$P(z) \to P(z_1, z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm} (\sqrt{2})^{n+m} z_1^n z_2^m .$$

Using this correspondence a lot of properties can be proved.

1. Convergence. If $P(z)$ is convergent at z_0 it is absolutely convergent at all the points in the disc $|z| < |z_0|$. This follows from an analogous property of holomorphic series in z_1 and z_2 which is called Abel's theorem.

There exists a positive real number r such that $P(z)$ converges absolutely in the disc $|z| < r$ and diverges for $|z| > r$. This number r is called the radius of convergence; r can be found as in one-dimensional holomorphic theory. Consider the sequence $\sqrt[n+m]{|a_{nm}|}$ and let l be its limit superior. Then $r = 1/l$. And this can be extended, of course, on series in the neighbourhood of a point $a \neq 0$ which can be done by putting $\zeta = z-a$ and also in the neighbourhood of the point $z = \infty$ by putting $\zeta = 1/z$. A series of the form $Q(z) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_{nm} z^n \bar{z}^m$ is called Laurent series. This expression means

$$Q(z) = \sum_{n,m=0}^{\infty} a_{nm} z^n \bar{z}^m + \sum_{\substack{n,m=0) \\ (n,m)\neq(0,0)}}^{\infty} a_{-n,-m} \frac{1}{z^n \bar{z}^m} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{-n,m} \frac{\bar{z}}{z^n} + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{n,-m} \frac{z^n}{\bar{z}^m}.$$

All these four parts of $Q(z)$ must be convergent, if Q should have any sense. The first part converges in a disc, the second part converges outside another disc. It is easy to prove that part three and part four converge in $r_{\mathrm{III}} < |z| < R_{\mathrm{III}}$, $r_{\mathrm{IV}} < |z| < R_{\mathrm{IV}}$, resp. Only if these four sets of convergence have a non-empty intersection, we can write the above expression for $Q(z)$.

The operations on series such as sum, difference, product, quotient, and the derivative of a power series, can be defined as in the two-dimensional case by restriction to the plane $z_1 = \bar{z}_2$.

2. Coefficient comparison. Next we come to a problem which can be also tackled by two-dimensional theory, that is the theorem of identity or, in other words, the coefficient comparison. At first we can state the following trivial theorem: *Let $P_1(z)$ and $P_2(z)$ be two given convergent power series. If P_1 and P_2 coincide at every point of an open subset of the common set of convergence, then P_1 and P_2 are identical, and we can equate coefficients of corresponding powers of z and $\bar{z}$.*

In one-dimensional holomorphic theory we have the following identity theorem: *Two series are identical if they coincide at every point of a convergent sequence of points.* We know that such a kind of identity theorem does not take place in two dimensional theory, we can find counter examples e.g. in Osgood's "Funktionentheorie".

H. Hornich (Monatshefte fur Mathematik, vol. 71 (1967), 214-217) gave necessary and sufficient conditions for convergent point sequences, so that two holomorphic functions $f(z_1, z_2)$ and $g(z_1, z_2)$ are identical if they coincide at every point of such a sequence. We can transfer this theorem, of course, to our series in z and $\bar z$.

The present author succeeded in proving a sufficient condition, from which it follows that the zeros of a series $P(z)$ cannot accumulate, and this gives us an identity theorem for series in z and $\bar z$, which is similar to the form well known from one-dimensional holomorphic theory.

3. Analytic continuation. In holomorphic theory the identity theorem leads to analytic continuation and the same is true here. Let us consider the series $P_a(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm}(z-a)^n(\bar z-\bar a)^m$. Let b be a point in the disc $|z-a| < r$, where r is the convergence radius. If the function P_a has another representation, say $P_b(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a'_{nm}(z-b)^n(\bar z-\bar b)^m$, where the coefficients a'_{nm} can be determined from the original series, it is possible to prove that P_b converges at least in $|z-b| < r-|b-a|$.

Like in holomorphic theory we use the following definitions. If the discs of convergence of two power series $P_a(z)$ and $P_b(z)$ have a non-empty intersection ω and P_a and P_b coincide in ω P_a is said to be a direct analytic continuation of P_b and conversely, P_b is a continuation of P_a.

The quantity consisting of all pairs $(z_0, P_0(z))$, where $P_0(z) = \sum_{n=0}^{\infty}\sum_{n=m}^{\infty} a^{(0)}_{nm} \times$ $\times (z-z_0)^n(\bar z-\bar z_0)^m$ which can be obtained directly or indirectly by an analytic continuation from a given series $P_a(z)$, is called a monogenic system of power series or a monogenic function and is denoted by $f(z)$.

The series P_a is called a primitive element. Each particular series of such a system $f(z)$ is called an elementary branch and in its of convergence disc K_0 it represents a single-valued branch of a monogenic function.

Every interior point of K_0 is called a regular point. A point $s \in \partial K_0$ is called regular if there exists a direct continuation $P_s(z)$ whose radius of convergence does not vanish. Otherwise it is called singular.

All these definitions are well known from holomorphic theory and they all can be transferred to the series considered here. Let us mention the following difference. In the holomorphic theory we have the theorem of natural boundary which means that at least one point of the boundary of the disc of convergence K_0 of an elementary branch P_0 is singular. This theorem does not take place here.

4. Isolated singularities. We restrict ourselves to isolated singular points. Let us consider a single-valued branch $f(z)$ of a monogenic function defined in an open connected set G'. Let c be an isolated boundary point of G' so that the union $G = \{G' \cup c\}$ is also an open connected set, then c is said to be an isolated singularity of $f(z)$.

Like in the holomorphic theory an isolated singular point can be of any one of three types:

i) $f(z)$ is bounded — c is called removable singularity;

ii) $f(z)$ is unbounded but $1/f$ is bounded — c is said to be a pole;

iii) $f(z)$ is unbounded and $1/f$ is unbounded, too — c is an essential singularity.

In studying non-holomorphic series in z and $\bar{z}$ we have to distinguish two types of removable points and poles.

Let $f(z)$ be bounded and let $\{z_n\} \to c$ be any convergent sequence of points. Then it is possible to choose a subsequence so that f tends to a finite value, a finite asymptotic value. Such a kind of removable singularity is called weak-removable. An isolated singular point c such that $f(z)$ can be defined or redefined in such a way as to be at least continuously at c is said to be strong-removable. And so a pole is said to be a weak one if $1/f$ is weak-removable and it is said to be a strong pole if $1/f$ is strong-removable.

The well-known Riemann's theorem on removable singularities can be generalized for strong-removable singularities, if the derivative $\partial_{\bar{z}} f$ is of $L^p(G)$, $p > 2$.

A generalization of Casorati Weierstrass' theorem on essential singularities takes also place if $\partial_{\bar{z}} f \in L^p$, $p > 2$.

5. Examples. At last we give some examples of series-expansion of functions in the neighbourhood of a singular point. We only mention the properties but we do not prove them.

Let $Q(z) = \sum\limits_{n=-\infty} \sum\limits_{m=-\infty} a_{nm} z \bar{z}^m$ be convergent in $0 < |z| < r$; $z = 0$, the origin is an isolated singularity. In order to get a better image we arrange the coefficients in a matrix

$$\left(\begin{array}{c|c} (-n, -m) & (n, -m) \\ \hline (-n, m) & (n, m) \end{array} \right)$$

So we have the following types of series:

i) $Q_1(z) = \sum\limits_{n=0}^{\infty} \sum\limits_{m=-\infty}^{n} a_{n-m,m} z^{n-m} \bar{z}^m$, that is Q_1 has a weak-removable singularity;

ii) $Q_2(z) = \sum\limits_{n=k}^{\infty} \sum\limits_{m=-\infty}^{n} a_{n-m,m} z^{n-m} \bar{z}^m$, i.e. Q_2 has a strong-removable singularity and $Q_2 \in C^{k-1}$;

iii) $Q_3(z) = \sum_{n=-n_1}^{\infty} \sum_{m=-n_2}^{\infty} a_{nm} z^n \bar{z}^m$ with $a_{-n_1,-n_2} \neq 0$, i.e. Q_3 has a strong pole;

iv) $Q_4(z) = \sum_{n=-n_0}^{\infty} \sum_{m=-\infty}^{n} a_{n-m,m} z^{n-m} \bar{z}^m$, here all types of singularities are possible. To decide which type of singularity is in question we have to investigate some of so called characteristic functions which are Fourier-series of the form

$$\chi_n(\varphi) = \sum_{m=-\infty}^{\infty} a_{n-m,m} e^{-2(m+k_n)i\varphi}.$$

More detailed considerations will be given in Mathematische Nachrichten. My works on these subjects will be published in 1971, Vol. 47, 49, and 51.

STRESZCZENIE

Tematem odczytu jest przedstawienie niektórych podstawowych własności szeregów potęgowych postaci $\sum a_{nm} z^n \bar{z}^m$. W szczególności, Autor zwraca uwagę na pewne analogie, ale też i na pewne różnice w zachowaniu się tych szeregów w porównaniu z szeregami potęgowymi postaci $\sum a_n z^n$, bądź $\sum a_{nm} z_1^n z_2^m$.

РЕЗЮМЕ

Тема работы — представление некоторых главных свойств степенных рядов вида $\sum a_{nm} z^n \bar{z}^m$. Особенное внимание обращено на некоторые аналогии и некоторые разницы в сохранении этих рядов по сравнению со степенными рядами вида $\sum a_n z^n$, $\sum a_{mn} z_1^m z_2^n$.