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A Certain Minimum Problem in the Class S

Pewien problem minimum w klasie S

Некоторая проблема минимум в классе S

Introduction. Let S be the class of all holomorphic and univalent functions $f(z) = z + a_2 z^2 + \ldots$ in the unit disc $|z| < 1$. Each function $f(z) \in S$ maps conformally the disc $|z| < 1$ onto a domain D which contains the origin 0 of the coordinate system and $w = 1/f(z)$ maps $|z| < 1$ onto a domain G, where G is the exterior of a continuum E of capacity 1. The origin 0 is contained in E. To each function $f(z) \in S$ corresponds in one-to-one way a continuum E, $\operatorname{cap} E = 1$ and the coordinate system with the origin $0 \in E$. When E is fixed, $\operatorname{cap} E = 1$, $0 \in E$, then the rotation round 0 gives a new function in S but the absolute values of the coefficients a_n, $n = 2, 3, \ldots$ remain unchanged. Therefore to each continuum E, $\operatorname{cap} E = 1$, there corresponds a certain subclass of the class S. In particular when E is a segment $\bar{E}$ of the length 4 we denote by $\bar{S}$ the corresponding subclass of S.

Let us consider the class S_c of all those functions $f(z) \in S$ which have the same positive value of the second coefficient a_2, i.e. $a_2 = c$, $c \in [0, 2]$. We shall prove the following

Theorem. *When $c > 0$ is close enough to 2, then the minimum value of $|a_3|, |a_4|, |a_5|$ of functions $f(z) \in S_c$ is assumed by function $\bar{f}(z) \in \bar{S}$, $\bar{f}(z) = z + cz^2 + (c^2-1)z^3 + (c^3-2c)z^4 + (c^4-3c^2+1)z^5 + \ldots$. The function $1/\bar{f}(z)$ gives the conformal mapping of the unit disc onto the exterior of the segment $\bar{E}$, the origin O of the coordinate system lies in the distance c from the middle $\bar{O}$ of $\bar{E}$ and the positive real axis has the direction of $\bar{O}O$.*

Auxiliary formulas. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the n^{th} extremal system of points in E, i.e. a system of n points in E such that $\prod_{j<k} |\eta_j - \eta_k|$

$= \sup_{w_j \in E} \prod_{j<k} |w_j - w_k|$. F. Leja proved in [3] the existence of the following limits

$$s_k = \lim_{n\to\infty} (\eta_1^k + \eta_2^k + \ldots + \eta_n^k)/n, \quad k = 1, 2, \ldots$$

The point $s_1 = O\overline{O}$ is the center of gravity of the natural mass distribution on E and its position relative to E remains unchanged after rotation or translation of the coordinate system.

Let us consider the set of all continua E of capacity 1 situated so that the center of gravity $\overline{O}$ is the common point for all E. It is known that all E lie inside the disc K of radius 2 centred at the point $\overline{O}$.

Among all E under consideration there are segments of length 4 which have their endpoints on the circumference of K, all other E have a positive distance from the boundary of K.

F. Leja [3] gave formulas which express the coefficients a_n, $n = 2, 3, \ldots$ of $f(z) \in S$ as polynomials in $s_1, s_2, s_3, \ldots$. If one computes the "moments" s_k relatively to the point $\overline{O}$ instead of the origin 0 one obtains

$$s_k = s_k(\overline{0}) + \binom{k}{1} s_1 s_{k-1}(\overline{0}) + \binom{k}{2} s_1^2 s_{k-2}(\overline{0}) + \ldots + s_1^k$$

The formulas given by F. Leja are the following

$$a_2 = -s_1, \; a_3 = a_2^2 - s_2(\overline{0})/2, \; a_4 = a_2^3 - a_2 s_2(\overline{0}) - s_3(\overline{0})/3,$$

$$a_5 = a_2^4 - 3a_2^2 s_2(\overline{0})/2 - 2a_2 s_3(\overline{0})/3 - s_4(\overline{0})/4 + 5s_2^2(\overline{0})/8.$$

As the rotation of the coordinate system does not change the modulus of a_n we can choose it so that the real axis has the direction of $\overline{O}O$ i.e. $a_2 = \overline{O}O \geqslant 0$.

For the Koebe function $z/(1-z)^2$ is $a_2 = 2$, $s_2(\overline{0}) = 2$, $s_3(\overline{0}) = 0$, $s_4(\overline{0}) = 6$. In the general case

$$\begin{aligned} a_2 &= 2 - \varepsilon + i\varepsilon_1, & \varepsilon &\geqslant 0 \\ s_2(\overline{0}) &= 2 - \delta + i\delta_1 & \delta &\geqslant 0 \\ s_4(\overline{0}) &= 6 - \eta + i\eta_1, & \eta &\geqslant 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \operatorname{re} a_3 &= 3 - 4\varepsilon + \delta/2 + \varepsilon^2 - \varepsilon_1^2 \\ \operatorname{re} a_4 &= 4 - 10\varepsilon + 2\delta - \operatorname{re} s_3(\overline{0})/3 + 6\varepsilon^2 - 6\varepsilon_1^2 - \varepsilon^3 + 3\varepsilon\varepsilon_1^2 + \varepsilon_1\delta_1 - \varepsilon\delta \\ \operatorname{re} a_5 &= 5 - 20\varepsilon + 7\delta/2 + \eta/4 - 4\operatorname{re} s_3(\overline{0})/3 + 5\delta^2/8 - 5\delta_1^2/8 + 21\varepsilon^2 - \\ &\quad - 21\varepsilon_1^2 + 2\varepsilon_1 \operatorname{im} s_3(\overline{0})/3 - 8\varepsilon^3 + \varepsilon^4 + 3\varepsilon^2\delta/2 + 6\varepsilon_1\delta_1 + 24\varepsilon\varepsilon_1^2 - \\ &\quad - 6\varepsilon^2\varepsilon_1^2 + \varepsilon_1^4 - 3\varepsilon_1^2\delta/2 - 3\varepsilon\varepsilon_1\delta_1 - 6\varepsilon\delta + 2\varepsilon \operatorname{re} s_3(\overline{0})/3 \end{aligned} \tag{2}$$

Proof of the theorem. As $a_2 = c > 0$ we obtain

$$\operatorname{re} a_3 = 3 - 4\varepsilon + \delta/2 + \varepsilon^2$$

Let us denote by $\bar{a}_n, n = 3, 4, \ldots$ the coefficients of the function $\bar{f}(z)$ then $\bar{a}_3 = \operatorname{re}\bar{a}_3 = 3 - 4\varepsilon + \varepsilon^2$. Hence

$$\operatorname{re} a_3 - \operatorname{re}\bar{a}_3 = \delta/2 \geqslant 0$$

for all functions in S_c. But $|a_3| \geqslant \operatorname{re} a_3$. Therefore $|a_3| \geqslant \bar{a}_3 = c^2 - 1$.

Similarly

$$\operatorname{re} a_4 - \operatorname{re}\bar{a}_4 = 2\delta - \operatorname{re} s_3(\bar{0})/3 - \varepsilon\delta, \quad \text{and} \quad \bar{a}_4 = \operatorname{re}\bar{a}_4.$$

One of Grunsky's inequalitees, see [1] has the form

(3) $$|s_2(\bar{0})\xi_1^2/2 + 4s_3(\bar{0})\xi_1\xi_2/3 + [s_4(\bar{0}) - s_2^2(\bar{0})\xi_2^2| \leqslant |\xi_1|^2 + 2|\xi_2|^2$$

where ξ_1 and ξ_2 are arbitrary numbers and the equality holds only for the Koebe function. Taking the real part of both sides for $\xi_1 = 2$ and $\xi_2 = 1/2$ we obtain using the previous notations

$$-2\delta + 4\operatorname{re} s_3(\bar{0})/3 - \eta/4 + \delta - \delta^2/4 + \delta_1^2/4 < 0.$$

Hence

(4) $$-\operatorname{re} s_3(\bar{0})/3 > -\eta/16 - \delta/4 + \delta_1^2/16 - \delta^2/16.$$

On the other hand it was proved in [2]

$$|3s_2^2(\bar{0})/4 - s_4(\bar{0})/4| \leqslant 3/2 \quad \text{for all} \quad f(z) \in S.$$

Therefore

(5) $$-3\delta - 3\delta_1^2/4 + 3\delta^2/4 + \eta/4 < 0$$

and

$$\eta/4 < 3\delta + 3\delta_1^2/4 - 3\delta^2/4.$$

From (4) and (5)

$$-\operatorname{re} s_3(\bar{0})/3 > -\delta - \delta_1^2/8 + \delta^2/8.$$

Hence

$$\operatorname{re} a_4 - \operatorname{re}\bar{a}_4 > \delta - \varepsilon\delta + \delta^2/8 - \delta_1^2/8.$$

But $\delta_1^2 + (2-\delta)^2 = |s_2(\bar{0})|^2 \leqslant 4$ for all $f(z) \in S$, see [1]. From the last formula follows

(6) $$\delta_1^2/8 = (\delta - b)/2 + (b^2 - \delta^2)/8 \leqslant (\delta - b)/2.$$

Hence

$$\operatorname{re} a_4 - \operatorname{re}\bar{a}_4 \geqslant \delta - \varepsilon\delta + \delta^2/8 - \delta/2 + b/2 > \delta[\tfrac{1}{2} - \varepsilon] > 0$$

for sufficiently small $\varepsilon > 0$, i.e. for $c = 2 - \varepsilon$ sufficiently close to 2. As $|a_4| \geqslant \operatorname{re} a_4$ and $\operatorname{re}\bar{a}_4 = \bar{a}_4$ it follows $|a_4| \geqslant \bar{a}_4 = c^3 - 2c$ for c close enough to 2.

We have (see (2))

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 = 7\delta/2 + \eta/4 - 4 \operatorname{re} s_3(\bar{0}) - 5\delta_1^2/8 + 5\delta^2/8 + $$
$$+ 2\varepsilon \operatorname{re} s_3(\bar{0})/3 + 3\varepsilon^2 \delta/2 - 6\varepsilon\delta.$$

According to (4)

$$\eta/4 - 4 \operatorname{re} s_3(\bar{0})/3 > -\delta - \delta^2/4 + \delta^2/4.$$

Therefore, see (6)

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > 5\delta/2 - 3\delta_1^2/8 + 3\delta^2/8 + 2\varepsilon \operatorname{re} s_3(\bar{0})/3 + 3\varepsilon^2 \delta/2 - 6\varepsilon\delta$$
$$> \delta + 3b/2 + 3\delta^2/8 + 3\varepsilon^2 \delta/2 - 6\varepsilon\delta + 2 \operatorname{re} s_3(\bar{0})/3$$
$$> \delta(1-6\varepsilon] + 2\varepsilon \operatorname{re} s_3(\bar{0})/3.$$

If we put in (3) $\xi_1 = -2$, $\xi_2 = 1/2$ we obtain

$$\operatorname{re} s_3(\bar{0})/3 > -\eta/16 - \delta/4 - \delta^2/16 + \delta_1^2/16$$

and multiplying by $2\varepsilon > 0$

$$2\varepsilon \operatorname{re} s_3(\bar{0})/3 > -\eta\varepsilon/8 - \delta\varepsilon/2 - \delta^2 \varepsilon/8 + \delta_1^2 \varepsilon/8.$$

Using (5) we obtain

$$2\varepsilon \operatorname{re} s_3(\bar{0})/3 > -2\varepsilon\delta - \delta_1^2 \varepsilon/4 + \delta^2 \varepsilon/4 > -3\varepsilon\delta + \varepsilon b + \delta^2 \varepsilon/4.$$

Hence $\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > \delta(1-9\varepsilon)$ and for sufficiently small $\varepsilon > 0$ $\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > 0$. As $|a_5| \geqslant \operatorname{re} a_5$ and $\bar{a}_5 = \operatorname{re} \bar{a}_5$ it follows $|a_5| \geqslant \bar{a}_5 = c^4 - 3c^2 + 1$ for c sufficiently close to 2.

Remark. From (2) follows immediately

$$(\bar{a}_4 - 4) - (\bar{a}_3 - 3) = -6\varepsilon + 5\varepsilon^2 - \varepsilon^2 < 0$$

for sufficiently small $\varepsilon > 0$ and

$$(\bar{a}_5 - 5) - (\bar{a}_4 - 4) = -10\varepsilon + 15\varepsilon^2 - 7\varepsilon^3 + \varepsilon^4 < 0$$

for $\varepsilon > 0$ close enough to 0.

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STRESZCZENIE

Autor zajmuje się klasą S_c funkcji $f(z) = z + cz^2 + \ldots$, analitycznych i jednolistnych w kole jednostkowym, przy czym $0 \leqslant c \leqslant 2$. Znajduje dokładne wartości $\min|f^{(k)}(0)/k!|$, $k = 3, 4, 5$, dla c bliskich 2.

РЕЗЮМЕ

Автор занимается классом S_c аналитических и однолистных функций $f(z) = z + cz^2 + \ldots$ в единичной окружности, при этом $0 \leqslant c \leqslant 2$. Получает точную оценку $\min|f^{(k)}(0)/k!|$ для c близких 2.