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DIETER GAIER

Entire Functions with Gap Power Series

Funkcje całkowite z lukowym szeregiem potęgowym

Целые функции с лакунарным степенным рядом

In this talk I shall present some results on entire functions whose power series expansion at 0 has gaps, which have been obtained some time ago [1], but which I thought would be best suitable to report at this conference.

1. Motivation. Assume that a series $\sum a_n$ with partial sums s_n is given. We say that $\sum a_n$ is Borel-summable to s, i.e. $B-\sum a_n = s$, if

$$e^{-z}\sum \frac{s_n z^n}{n!} \to s \quad (z = x \to +\infty).$$

Several authors, Pitt, Erdös, Meyer-König, Zeller, and recently Melnik have considered the problem:

$$B-\sum a_n = s, \quad a_n = 0 \text{ for "many" } n \Rightarrow: \sum a_n = s,$$

which is the so called high indices theorem for Borel summability.

The most difficult part in results of this type is to obtain first some order condition on the a_n, for example $a_n = O(K^n)$ $(n \to \infty)$ for some $K < \infty$, and this is where complex variable methods can be applied successfully.

Namely, the hypothesis $B-\sum a_n = s$ implies that $e^{-x}\sum \frac{a_n x^n}{n!} \to 0$ $(x \to +\infty)$, and the order condition $a_n = O(K^n)$ is equivalent to saying that $\sum \frac{a_n z^n}{n!}$ is entire and of exponential type. So our problem is transformed into the following

Problem: If $f(z) = \sum a_n z^n$ is entire, and $a_n = 0$ for "many" n, and if further $f(x) = O(e^x)$ $(x \to +\infty)$, show that f is of exponential type.

The gap condition permits us to conclude from the radial growth of f to the growth of f in the plane.

2. Theorem. More precisely, we shall prove the following

Theorem. *Let* $f(z) = \sum a_n z^n$ *be entire, and* $a_n = 0$ *for* $n \neq \lambda_k$ *with* $\sum \lambda_k^{-1} < \infty$. *If then* $f(x) = O\big(\exp(x^a)\big)$ $(x \to +\infty)$ *for some* $a > 0$, *then* f *is at most of order* a, *type* 1.

We remark that the condition on the $\{\lambda_k\}$ is best possible. According to Macintyre there is, to any given $\{\lambda_k\}$ with $\sum \lambda_k^{-1} = \infty$, an entire function f of infinite order which is bounded for $z = x > 0$.

3. Proof of theorem. With V. Bernstein we consider for arbitrary fixed $T < \infty$ the transform

$$H(z) = \int_0^T f(t) t^{-z-1} dt.$$

We assume, as we may, $a_0 = 0$, so that the integral converges for $\operatorname{Re} z < 1$, and H is a regular function in the halfplane $\operatorname{Re} z < 1$.

To obtain its analytic continuation beyond $\operatorname{Re} z = 1$, we write in $\operatorname{Re} z \leqslant 0$

$$H(z) = \int_0^T \sum a_n t^{n-z-1} dt = \sum a_n \int_0^T t^{n-z-1} dt$$

$$= \sum a_n \frac{T^{n-z}}{n-z} = -T^{-z} \sum a_n \frac{T^n}{z-n}.$$

The right hand side is a meromorphic function in the plane, and thus it represents the analytic continuation of H beyond $\operatorname{Re} z = 1$. Possible singularities are simple poles at $z = n$ with residues $-a_n$ $(n = 1, 2, \ldots)$. Thus we transformed the gap condition: Many a_n are zero, to a complex analytic condition: H has few poles.

We also notice that on the imaginary axis, $z = iy$,

$$|H(z)| \leqslant \int_0^T |f(t)| t^{-1} dt \leqslant C_1 \exp(T^a)/T^a \qquad (T \geqslant 1).$$

Finally, if we stay away from the poles of H, $|z-n| \geqslant \eta > 0$, we have

$$|H(z)| \leqslant T^{-x} \frac{C_2(T)}{\eta}.$$

In order to remove the poles, we consider the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z} = \prod_{k=1}^{\infty} \left(1 - \frac{2z}{\lambda_k + z}\right)$$

and the new function

$$\varphi(z) = H(z)B(z) \text{ in } \operatorname{Re} z \geqslant 0.$$

It is regular in $\operatorname{Re} z \geqslant 0$, and on the imaginary axis

$$|\varphi(z)| = |H(z)| \leqslant C_1 \exp(T^a)/T^a.$$

Furthermore, the estimate of H outside the poles gives

$$|\varphi(z)| \leqslant C_3(T)T^{-x} \quad \text{for} \quad \operatorname{Re} z = x \geqslant 0;$$

in particular, φ is bounded in the positive half plane.

Now we apply a theorem of Phragmén-Lindelöf type:

If f is regular and of exponential type in $\operatorname{Re} z \geqslant 0$, $|f(iy)| \leqslant M$, *and* $\limsup_{x\to+\infty} \frac{\log|f(x)|}{x} \leqslant C$, *then* $|f(z)| \leqslant Me^{Cx}$ $(z = x+iy, x \geqslant 0)$.

Applied to φ, we obtain

$$|\varphi(z)| \leqslant C_1 \frac{\exp(T^a)}{T^a} T^{-x} \quad (z = x+iy, x \geqslant 0),$$

and if we put $z = n$ for $n = \lambda_k$, this gives

$$|a_n|\,|B'(\lambda_k)| \leqslant C_1 \frac{\exp(T^a)}{T^a} T^{-n},$$

where the right hand side is

$$O(1)\left[\sqrt{n}\,\Gamma\left(1+\frac{n}{a}\right)\right]^{-1},$$

if $T > 0$ is chosen so that $T^a = 1 + \frac{n}{a}$. A more detailed estimate (see [1], p. 252) now shows that $|B'(\lambda_k)|^{-1} = O(1)e^{\varepsilon\lambda_k}$ for every $\varepsilon > 0$. Putting everything together, we arrive at

$$a_n = O(1)\frac{e^{\varepsilon n}}{\Gamma\left(1+\frac{n}{a}\right)} \quad (n \to \infty),$$

valid for every $\varepsilon > 0$, which proves our theorem.

We may remark that in the case of larger gaps

$$\lambda_{k+1}-\lambda_k \geqslant \theta\sqrt{\lambda_k} \quad (\theta > 0 \text{ fixed}),$$

our method gives (if $a = 1$)

$$a_n = O(1)e^{\varepsilon\sqrt{n}}\frac{\sqrt{n}}{n!} \quad (n \to \infty)$$

for every $\varepsilon > 0$.

In conclusion, we point out that our method was modified by Halász [2] to give a complex variable proof of the high indices theorem of Hardy-Littlewood and some refinements of it.

REFERENCES

[1] Gaier, D., *On the coefficients and the growth of gap power series*, J. SIAM Numer. Analysis 3 (1966), 248–265.

[2] Halász, G., *Remarks to a paper of D. Gaier on gap theorems*, Acta Sci. Math. (Szeged) 28 (1967), 311–322.

Added in proof:

Further generalizations of our theorem have been obtained by Anderson and Binmore in Trans. Amer. Math. Soc. 161 (1971), 381–400.

STRESZCZENIE

Jak wykazał Macintyre w r. 1952, funkcja całkowita f, której szereg Taylora ze środkiem w zerze ma luki Fabry'ego i która jest ograniczona dla $z = x > 0$, redukuje się do stałej.

Twierdzenie to można uogólnić osłabiając ograniczenie wzrostu. Np. jeśli f ma luki Fabry'ego oraz $f(x) = O(\exp x^a)$ $(x \to +\infty)$ dla pewnego $a > 0$, to f jest rzędu conajwyżej a i typu 1. Przypadek $a = 1$ pozwala na otrzymanie dowodu twierdzenia o dużych wskaźnikach dla sumowalności borelowskiej.

РЕЗЮМЕ

Как доказано Макинтайром, целая функция, которая имеет степенной ряд в начале координат с лакунами Фабри и которая ограничена для $z = x > 0$, редуцирует к константе.

Эта теорема допускает следующее обобщение.

Пусть f — целая функция, $f(x) = O(\exp x^a)$ $(x \to +\infty)$ для некоторого $a > 0$ и степенной ряд имеет лакуны Фабри, тогда f конечного порядка небольше a и типа 1. Случай $a = 1$ дает доказательство теоремы о больших индексах для борелевской суммируемости.