A N N A L E S
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXII/XXIII/XXIV, 8 SECTIO A 1968/1969/1970

Instytut Matematyki, Politechnika Łódzka, Łódź

IZYDOR DZIUBIŃSKI

Variational Formulas for Quasi-Starlike and Quasi-Convex Functions

Wzory wariacyjne dla funkcji quasi-gwiaździstych i quasi-wypukłych

Вариационные формулы для квази-звездных и квази-выпуклых функций

Let $\mathcal{G}^\lambda$ denote the class of quasi-starlike functions $g(z)$ determined by the equation

(1) $$F(g(z)) = \lambda F(z), \quad |z| < 1,$$

where $F(z)$ is a starlike function and λ fixed, $0 < \lambda < 1$.

Further, let $\mathcal{G}^\lambda_m$ be the subclass of functions $g(z)$ of $\mathcal{G}^\lambda$ determined by the equation $G(g(z)) = \lambda G(z)$, where $G(\zeta)$ is of the form

$$G(\zeta) = \frac{\zeta}{\prod_{k=1}^{m} (1-\sigma_k\zeta)^{\beta_k}}; \quad \sum_{k=1}^{m} \beta_k = 2; \ \beta_k > 0,$$

$$|\sigma_k| = 1, \quad \sigma_i \neq \sigma_j \quad \text{for} \quad i \neq j, \quad i, j, k = 1, \ldots, m.$$

Finally, let $\mathcal{W}^M$ denote the class of quasi-convex functions $h(z)$ determined by the equation

(2) $$f(h(z)) = \lambda f(z), \quad |z| < 1,$$

where $f(z)$ is a convex function and λ fixed, $0 < \lambda < 1$, and $\mathcal{W}^\lambda_m$ is analogous to the subclass $\mathcal{G}^\lambda_m$ of the class $\mathcal{W}^\lambda$.

Using the variational formulas for starlike functions we shall derive analogous formulas for functions of the class $\mathcal{G}^\lambda$ and $\mathcal{W}^\lambda$, resp.

It is well known that for all sufficiently small positive ε there exist functions of the form

(3) $$F_\varepsilon(z) = F(z) + \varepsilon F(z) Q(z, a) + o(\varepsilon),$$

where

$$Q(z,a) = AK(z,a) - \bar{A}K\left(z,\frac{1}{\bar{a}}\right) - \frac{A}{H(a)}L(z,a) - \frac{\bar{A}}{\overline{H(a)}}L\left(z,\frac{1}{\bar{a}}\right),$$

(4)

$$K(z,a) = \frac{z+a}{z-a} + H(z),$$

$$L(z,a) = \frac{z+a}{z-a}H(z) + 1,$$

$$H(z) = \frac{zF'(z)}{F(z)}, \quad |a| < 1, \quad A \text{ — any complex number.}$$

and the term $o(\varepsilon)/\varepsilon$ tends to zero uniformly on compact subset of the unit disk as $\varepsilon \to 0$, which also belong to S^*.

We can get easily from the formula (3) (compare e.g. [4]) a variational formula for functions inverse to the functions of the class S^* in the form

(5) $$F_\varepsilon^{-1}(w) = F^{-1}(w) - \varepsilon w Q\big(F^{-1}(w), a\big)\big(F^{-1}(w)\big)' + o(\varepsilon),$$

where

$$w = F(z) \in F\big(K(0,1)\big).$$

By (5) and by the starshapedness of F we have

$$F_\varepsilon^{-1}\big(\lambda F(z)\big) = F^{-1}\big(\lambda F(z)\big) - \varepsilon\lambda F(z) Q\big(F^{-1}(\lambda F(z)), a\big) F^{-1\prime}\big(\lambda F(z)\big) + o(\varepsilon);$$

hence, by (1) we get the formula

(6) $$F_\varepsilon^{-1}\big(\lambda F(z)\big) = g(z) - \varepsilon \frac{F(z)}{F'(z)} Q\big(g(z), a\big) g'(z) + o(\varepsilon).$$

On the other hand, from (3) we conclude that

(7) $$F_\varepsilon^{-1}\big(\lambda F_\varepsilon(z)\big) = F_\varepsilon^{-1}\big(\lambda F(z) + \varepsilon\lambda F(z) Q(z,a) + o(\varepsilon)\big),$$

therefore

(7′) $$F_\varepsilon^{-1}\big(\lambda F_\varepsilon(z)\big) = F_\varepsilon^{-1}\big(\lambda F(z)\big) + \varepsilon\lambda F(z) Q(z,a) F_\varepsilon^{-1\prime}\big(\lambda F(z)\big) + o(\varepsilon)$$

and finally after replacing $F_\varepsilon^{-1\prime}\big(\lambda F(z)\big)$ by $F^{-1\prime}\big(\lambda F(z)\big)$ in (7) the error will be $o(\varepsilon)$ and so

(8) $$F_\varepsilon^{-1}\big(\lambda F_\varepsilon(z)\big) = F_\varepsilon^{-1}\big(\lambda F(z)\big) + \varepsilon\lambda F(z) Q(z,a) F^{-1\prime}\big(\lambda F(z)\big) + o(\varepsilon).$$

From (8) together with (1) we obtain immediately

(9) $$F_\varepsilon^{-1}\big(\lambda F(z)\big) = g_\varepsilon(z) - \varepsilon \frac{F(z)}{F'(z)} Q(z,a) g'(z) + o(\varepsilon).$$

From the relations (7), (9) and (1) it is easy to deduce the following

Theorem 1. *If $g(z)$ belongs to the class $\mathscr{S}^\lambda$ then for all sufficiently small numbers ε there exist functions $g_\varepsilon(z)$ of the form*

$$g_\varepsilon(z) = g(z) + \varepsilon \frac{F(z)g'(z)}{F'(z)}\big(Q(z,a) - Q(g(z),a)\big) + o(\varepsilon)$$

belonging to the class $\mathscr{S}^\lambda$, where $Q(z,a)$ is the function determined by (4); $|a| < 1$ and A is an arbitrary complex number.

We shall be now concerned with the variational formula for the starlike convex functions. Integrating the left and right hand side of the equality (3) we easily obtain the following

Theorem 2. *If $f(z)$ is a convex functions of the class $\hat{S}$ then for all sufficiently small numbers ε there exist functions $f_\varepsilon(z)$ of the form*

$$w_\varepsilon = f_\varepsilon(z) = f(z) + \varepsilon \int_0^z f'(z)\hat{Q}(z,a)\,dz + o(\varepsilon) \tag{10}$$

belonging to the class $\hat{S}$, too, where

$$\hat{Q}(z,a) = A\hat{K}(z,a) - \bar{A}\hat{K}\Big(z, \frac{1}{\bar{a}}\Big) - \frac{A}{H(a)}\hat{L}(z,a) - \frac{\hat{A}}{H(a)}\hat{L}\Big(z, \frac{1}{\bar{a}}\Big),$$

$$\hat{K}(z,a) = \frac{z+a}{z-a} + \hat{H}(z),$$

(11)

$$\hat{L}(z,a) = \frac{z+a}{z-a}\hat{H}(z) + 1,$$

$$\hat{H}(z) = \frac{zf''(z)}{f'(z)} + 1,$$

$|a| < 1$, and A is an arbitrary complex number.

We conclude further from (10) that

$$f_\varepsilon^{-1}(w_\varepsilon) = f_\varepsilon^{-1}\Big(w + \varepsilon \int_0^{f^{-1}(w)} f'(z)\hat{Q}(z,a)\,dz + o(\varepsilon)\Big)$$

and after analogous calculations as before we obtain

$$f_\varepsilon^{-1}(w) = f^{-1}(w) - \varepsilon\big(f^{-1}(w)\big)' \int_0^{f^{-1}(w)} f'(z)\hat{Q}(z,a)\,dz + o(\varepsilon). \tag{12}$$

Putting $w = \lambda f(z)$ into (12) we have by (2):

$$f_\varepsilon^{-1}\big(\lambda f(z)\big) = h(z) - \varepsilon \frac{h'(z)}{\lambda f'(z)} \int_0^{h(z)} f'(z)\hat{Q}(z,a)\,dz + o(\varepsilon). \tag{13}$$

On the other hand one concludes from (10) that

$$f_\varepsilon^{-1}\big(\lambda f_\varepsilon(z)\big) = f_\varepsilon^{-1}\big(\lambda f(z)\big) - \varepsilon\lambda f^{-1\prime}\big(\lambda f(z)\big)\int_0^z f'(z)\hat{Q}(z,a)\,dz + o(\varepsilon)$$

and after analogous calculations as in (7), (7'), (8), (9) we conclude that

$$f_\varepsilon^{-1}\big(\lambda f(z)\big) = h_\varepsilon(z) - \varepsilon\frac{h'(z)}{f'(z)}\int_0^z f'(z)Q(z,a)\,dz + o(\varepsilon).$$

Using this and (13) we obtain immediately

$$(14)\quad h_\varepsilon(z) = h(z) + \varepsilon\frac{h'(z)}{f'(z)}\left[\int_0^z f'(z)\hat{Q}(z,a)\,dz - \frac{1}{\lambda}\int_0^{h(z)} f'(z)\hat{Q}(z,a)\,dz\right] + o(\varepsilon)$$

From (14) we obtain

Theorem 3. *If $h(z)$ belongs to the class $\mathscr{W}^M$ then for all sufficiently small numbers ε there exist functions $h_\varepsilon(z)$ of the form*

$$h_\varepsilon(z) = h(z) + \varepsilon\frac{h'(z)}{f'(z)}\int_0^z f'(z)\Big(\hat{Q}(z,a) - \hat{Q}\big(h(z),a\big)\Big)dz + o(\varepsilon)$$

which belong to the class $\mathscr{W}^M$.

The equations of the extremal functions with respect to functionals depending on a finite number of Taylor's coefficients at the point $z = 0$ can be now obtained for the classes $\mathscr{G}_m^\lambda$ and $\mathscr{W}_m^\lambda$ by means of Lagrange multipliers method. Also the case of functionals depending on the value of function and a finite number of derivatives at an arbitrary point of unit circle, can be settled.

By means of these equations the coefficient estimates for the functions of the class $\mathscr{G}^M$ and $\mathscr{W}^M$, as well as estimates of the modulus of the function argument in $\mathscr{G}^M$ may be gained. It seems that the obtained variational formulas can be applied in studying various extremal problems in the classes considered.

REFERENCES

[1] Александров И. А. *Вариация звездообразных функций*, Вопросы математики и механики, Томск, (1961), 61–71.

[2] Dziubiński, I., *Quasi-starlike functions*, Bull. Acad. Polon. Sci. Vol. XVI, No. 6 (1968), 477–479.

[3] Hummel, J. A., *A variational method for starlike functions*, Proc. Am. Math. Soc. 9 (1958), 82–87.

[4] Netanyahu, E., *Extremal problems for schlicht functions in the exterior of the unit circle*, Can. J. Math., Vol. 17 (1965), 335–341.

STRESZCZENIE

Autor podaje wzory wariacyjne dla wprowadzonych przez niego klas funkcji quasi-gwiaździstych i quasi-wypukłych.

РЕЗЮМЕ

Автор установил вариационные формулы для введенных им классов квази-звездных и квази-выпуклых функций.