ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXII/XXIII/XXIV, 7 SECTIO A 1968/1969/1970

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On Meromorphic Quasi-starlike Functions

O funkcjach quasi-gwiaździstych meromorficznych

О квази-звездных мероморфных функциях

Let Σ^* denote the class of functions

$$F(z) = \frac{1}{z} + A_0 + A_1 z + \ldots, \tag{1}$$

which are univalent and holomorphic in $|z| < 1$, except for a pole at $z = 0$, and map $|z| < 1$ onto a domain whose complement is starlike with respect to the origin.

The class of functions $f(z)$ determined by the equation

$$F\left(\frac{1}{f(z)}\right) = MF(z), \tag{2}$$

where $F \in \Sigma^*$ and M is fixed, $(1 < M < \infty)$, will be called the class of meromorphic quasi-starlike functions and denoted by Σ^M.

Let us introduce the following notations

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots, \ |z| < 1, \tag{3}$$

where $a_{-1} = M$, and

$$\left(\frac{1}{f(z)}\right)^k = a_k^{(k)} z^k + a_{k+1}^{(k)} z^{k+1} + \ldots, \quad k = 1, 2, \ldots. \tag{4}$$

Further, let $\Sigma_{(m)}^M$ denote the subclass of Σ^M of meromorphic quasi-starlike functions determined by the equation

$$\frac{1}{f(z)} \prod_{k=1}^{m} (f(z) - \sigma_k)^{\beta_k} = \frac{M}{z} \prod_{k=1}^{m} (1 - \sigma_k z)^{\beta_k}, \tag{5}$$

where

(6) $$\sigma_k = e^{i\varphi_k},\ \varphi_k - \text{real},\ \sigma_i \neq \sigma_k \text{ for } i \neq k,\ i, k = 1, \ldots, m$$

and

(7) $$\sum_{k=1}^{m} \beta_k = 2, \quad \beta_k > 0, \quad k = 1, 2, \ldots, m.$$

Suppose that

(8) $$H = H(z_0, z_1, \ldots, z_N, \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_N)$$

is a real-valued function of $2N+2$ complex variables, defined in an open and sufficiently large set V and suppose that $\operatorname{grad} H \neq 0$ at every point of V.

Given a function $f(z)$ of the form (3), let

(9) $$H_f = H(a_0, a_1, \ldots, a_N, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_N).$$

One can prove the following

Theorem 1. *If the functional* (9) *attains its extremal value for a function* $f(z)$ *of the class* $\Sigma^M_{(m)}$, *then this function satisfies the following equations:*

$$\frac{f'(z)\mathscr{L}\left(\dfrac{1}{f(z)}\right)}{f(z)\mathscr{R}\left(\dfrac{1}{f(z)}\right)} + \frac{1}{z}\frac{\mathscr{L}(z)}{\mathscr{R}(z)} = 0$$

and

$$\frac{f'(z)}{f(z)} \frac{\tilde{\mathscr{L}}\left(\dfrac{1}{f(z)}\right)}{\tilde{\mathscr{R}}\left(\dfrac{1}{f(z)}\right)} + \frac{1}{z}\frac{\tilde{\mathscr{L}}(z)}{\tilde{\mathscr{R}}(z)} = 0,$$

where

$$\mathscr{L}(z) = \sum_{k=1}^{N+1}\left(\frac{C_k}{z^k} + \bar{C}_k z^k\right) + C_0,$$

$$\mathscr{R}(z) = \sum_{k=1}^{N+1}\left(\frac{D_k}{z_k} - \bar{D}_k z^k\right),$$

$$\tilde{\mathscr{L}}(z) = \sum_{k=1}^{N+1}\left(\frac{E_k}{z^k} - \bar{E}_k z^k\right) + E_0,$$

$$\tilde{\mathscr{R}}(z) = \sum_{k=1}^{N+1}\frac{1}{k}\left(\frac{D_k}{z^k} + \bar{D}_k z^k\right) + D_0,$$

$$(11)\quad D_k = \sum_{l=k-1}^{N} H_l \sum_{n=0}^{l-k+1} B_{l-k-n+1}\big((n-1)a_{n-1} - a_l^{(l-n)}\big), \quad k = 1, \ldots, N+1,$$

$$(12)\quad D_0 = -\frac{1}{2}\sum_{k=1}^{N+1} \frac{1}{k}(D_k d_k + \overline{D}_k \overline{d}_k),$$

$$C_k = \sum_{l=k}^{N+1} D_l d_{l-k}, \quad k = 1, \ldots, N+1,$$

$$C_0 = \sum_{l=1}^{N+1} D_l d_l \quad \text{and} \quad C_0 = \overline{C}_0,$$

$$E_k = \sum_{l=k}^{N+1} \frac{1}{l} D_l d_{l-k}, \quad k = 1, \ldots, N+1,$$

$$E_0 = \frac{1}{2}\sum_{k=1}^{N+1} \frac{1}{k}(D_k d_k - \overline{D}_k \overline{d}_k),$$

$$(13)\quad H_k = \frac{\partial H}{\partial a_k} + \overline{\left(\frac{\partial H}{\partial \overline{a}_k}\right)}, \quad k = 0, 1, \ldots, N,$$

$$d_k = \sum_{l=1}^{m} \beta_l \sigma_l^k, \quad k = 1, 2, \ldots, d_0 = 1,$$

$$(15)\quad B_k = (-1)^k \begin{vmatrix} d_1 & d_0 & 0 & \ldots & 0 \\ d_2 & d_1 & d_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ d_k & d_{k-1} & d_{k-2} & \ldots & d_1 \end{vmatrix}, \quad k = 1, 2, \ldots, B_0 = 1,$$

Moreover, the numbers $\overline{\sigma}_t$, $t = 1, \ldots, m$, which appear in (5) *and* (6) *are roots of $\mathscr{R}(z)$ and double roots of $\overline{\mathscr{R}}(z)$.*

The proof is based on the method of Lagrange's multipliers for the functions of complex variables [1].

Theorem 2. *The extremal value of the functional H_f is attained in the class Σ^M, for a funcion which belongs to the class $\bigcup_{m=1}^{N+1} \Sigma^M_{(m)}$.*

Proof. Let H^* denote e.g. maximum of the functional H_f in Σ^M and let H_k^* denote an analogous maximum in $\bigcup_{m=1}^{k} \Sigma^M_{(m)}$, $k = 1, 2, \ldots$. Let us observe first, that $H_k^* = H_{N+1}^*$ for $k > N+1$ and consequently

$$(16)\quad \sup_k H_k^* = H_{N+1}^*.$$

In fact, because the class $\bigcup_{m=1}^{k} \Sigma_{(m)}^{M}$, $k = 1, 2, \ldots$, is compact and H is continuous, we can find a function $f \in \Sigma_{(i)}^{M}$, $i \leqslant k$, realizing the mentioned extremum. Then from Theorem 1 it follows that the function $\tilde{\mathscr{R}}(z)$ of the form (10) has double roots at points $z = \overline{\sigma}_t$, $t = 1, \ldots, i$. This implies $i \leqslant N+1$ and, as consequently, inequality (16) follows.

Next, we prove that

$$H^* \leqslant H^*_{N+1}. \tag{17}$$

It is not difficult to observe that we can approximate any meromorphic quasi-starlike function by the functions of the classes $\Sigma_{(m)}^{M}$ $(m = 1, 2, \ldots)$. Therefore, the assumption $H_{f_0} > H^*_{N+1}$ with $f_0 \in \Sigma^M$ contradicts (16) and this implies the inequality (17).

Finally, we can observe using the definitions of meromorphic quasi-starlike functions $f(z)$ and quasi-starlike functions that the function $g(z)$ of the form $g(z) = 1/f(z)$ is quasi-starlike. From this and from a result obtained by I. Dziubiński [3] the following result easily follows.

Theorem 3. *The necessary and sufficient condition for a function* $F(z) = \dfrac{1}{M} f(z)$, $f \in \Sigma_{(m)}^{M}$, *to belong to the class* Σ^* *is that the following conditions are satisfied*: $\beta_k = 2/m$, $k = 1, 2, \ldots, m$,

$$\sigma_k = e^{i\frac{2k\pi}{m}}, \text{ when } m \text{ is an odd number}, \ k = 1, 2, \ldots, m, \tag{18}$$

$$\sigma_k = e^{i\left(\frac{4\pi}{m}\left[\frac{k-1}{2}\right]+(-1)^{k-1}\varphi\right)}, \text{ when } m \text{ is an even number}, \ k = 1, 2, \ldots, m, \tag{19}$$

where φ *is an arbitrary real number*, $0 < \varphi < 2\pi/m$, *and the numbers* σ_k *are determined by* (18), (19) *up to a rotation.*

In the next part of this paper we shall give sharp estimates of coefficients a_0, a_1, a_2 of a meromorphic quasi-starlike function.

Let us consider the functional H_f of the form

$$H_f = \operatorname{re} a_n = \tfrac{1}{2}(a_n + \overline{a}_n), \quad n = 0, 1, 2, \tag{20}$$

and suppose, that

$$\operatorname{re} a_n = a_n > 0, \quad n = 0, 1, 2, \tag{21}$$

for the function $f^*(z)$ realizing an extremum of the functional (20).

1°. Let $n = 0$. From Theorem 2 it follows immediately that the functional (20) attains its extremal value for a function $f^*(z)$ belonging to the class $\Sigma_{(1)}^{M}$. Hence, $f^*(z)$ is determined by the equation (5) $(m = 1)$ and the coefficient a_0 of $f^*(z)$, in view of (21), has the form

$$a_0 = 2M\left(1 - \frac{1}{M}\right).$$

Therefore, for every function $f \in \Sigma^M$ we have

$$|a_0| \leqslant 2M\left(1-\frac{1}{M}\right). \tag{22}$$

2°. Let $n = 1$. Then $f^*(z)$ realizing an extremum of H_f belongs to the class $\bigcup_{k=1}^{2} \Sigma^M_{(k)}$.

a) At first we suppose, that $f^* \in \Sigma^M_{(1)}$. Then $f^*(z)$ is determined by the equation (5) ($m = 1$) and its coefficient $a_{1(1)}$, in view of (21), has the form

$$a_{1(1)} = M\left(1-\frac{1}{M^2}\right). \tag{23}$$

b) Next, we suppose that $f^* \in \Sigma^M_{(2)}$. Then $f^*(z)$ is given by the equation (5), ($m = 2$), and its coefficient $a_{1(2)}$ is of the form

$$a_{1(2)} = \frac{M^2-1}{M}[(\beta_1\sigma_1+\beta_2\sigma_2)^2-(\beta_1\sigma_1^2+\beta_2\sigma_2^2)]. \tag{24}$$

From Theorem 1 it follows that the numbers $\bar{\sigma}_t (t = 1, 2)$ are double roots of the function $\mathscr{R}(z)$ determined by (10)-(15). Hence, the numbers σ_1, σ_2, β_1, β_2 associated with $f^*(z)$ by the formula (5) satisfy the following system of the equations

$$\begin{cases} \sigma_1^2\sigma_1^2 = 1 \\ \sigma_1(1-\beta_1)+\sigma_2(1-\beta_2) = 0 \\ \beta_1+\beta_2 = 2. \end{cases}$$

Hence, we obtain

$$\sigma_1 = e^{i\varphi}, \quad \sigma_2 = e^{-i\varphi}, \quad \beta_1 = \beta_2 = 1, \tag{25}$$

or

$$\sigma_1 = e^{i\varphi}, \quad \sigma_2 = -e^{-i\varphi}, \quad \beta_1 = \beta_2 = 1. \tag{26}$$

Further, from (24)-(26) and (21) it follows that the coefficient $a_{1(2)}$ of $f^*(z)$ is of the form

$$a_{1(2)} = M\left(1-\frac{1}{M^2}\right). \tag{27}$$

We conclude from (23) and (27) that:
For every function $f \in \Sigma^M$ we have

$$|a_1| \leqslant M\left(1-\frac{1}{M^2}\right). \tag{28}$$

3°. Let $n = 2$. Then $f^*(z)$ realizing an extremum of H_f belongs to the class $\bigcup_{k=1}^{3} \Sigma_{(k)}^M$.

a) Suppose first that $f^* \in \Sigma_{(1)}^M$.

Then the coefficient $a_{2(1)}$ of $f^*(z)$ in view of (5), is given by the formula

$$a_{2(1)} = \frac{2}{M}\left(1 - \frac{1}{M}\right). \tag{29}$$

b) Next, suppose that $f^* \in \Sigma_{(2)}^M$.

Then, from the equation (5), ($m = 2$), it follows that its coefficient $a_{2(2)}$ is determined by the formula

$$\begin{aligned} a_{2(2)} = \frac{1-M}{6M^2} [& (M^2+M+4)(\beta_1\sigma_1+\beta_2\sigma_2)^3 - \\ & -3(M^2+M+2)(\beta_1\sigma_1+\beta_2\sigma_2)(\beta_1\sigma_1^2+\beta_2\sigma_2^2)+ \\ & +2(M^2+M+1)(\beta_1\sigma_1^3+\beta_2\sigma_2^3)]. \end{aligned} \tag{30}$$

From Theorem 1 it follows that the numbers $\bar{\sigma}_1, \bar{\sigma}_2$ are double roots of $\tilde{\mathscr{R}}(z)$ given by (10) and (11)-(15). Another roots of $\tilde{\mathscr{R}}(z)$ are denoted by $z_1 = \varrho_1 \bar{\sigma}, z_2 = \frac{1}{\varrho_1} \bar{\sigma}$, where $|\sigma| = 1, 0 < \varrho_1 \leqslant 1$. Hence it follows that $\sigma_1, \sigma_2, \beta_1, \beta_2$, associated with $f^*(z)$ by the formula (5) satisfy the following system of equations

$$\begin{cases} \beta_1+\beta_2 = 2 \\ \sigma_1^2\sigma_2^2\sigma^2 = 1 \\ \sigma_1+\sigma_2+\varrho\sigma = \dfrac{3}{4}(\beta_1\sigma_1+\beta_2\sigma_2)\dfrac{M^2+M+2}{M^2+M+1} \\ (\sigma_1+\sigma_2)^2+2\sigma_1\sigma_2+\sigma^2+4\varrho\sigma(\sigma_1+\sigma_2) = \\ = \dfrac{3}{2}\dfrac{M^2+M+4}{M^2+M+1}(\beta_1\sigma_1+\beta_2\sigma_2)^2 - \dfrac{3}{2}\dfrac{M^2+M+2}{M^2+M+1}(\beta_1\sigma_1^2+\beta_2\sigma_2^2), \end{cases} \tag{31}$$

where $\varrho = 1/2(\varrho_1 + 1/\varrho_1)$.

From system (31) we obtain

$$\sigma_1 = e^{-\frac{\pi}{3}i}, \quad \beta_1 = 1 - \frac{1}{2}\sqrt{\frac{2}{3}\frac{2M^2+2M+5}{M^2+M+4}}$$

$$\sigma_2 = -e^{-\frac{\pi}{3}i}, \quad \beta_2 = 1 + \frac{1}{2}\sqrt{\frac{2}{3}\frac{2M^2+2M+5}{M^2+M+4}}$$

and, as a consequence of this and the equation (30), we obtain

$$(32)\qquad a_{2(2)} = \frac{2}{9}\,\frac{(M-1)(2M^2+2M+5)}{M^2}\sqrt{\frac{2}{3}\,\frac{2M^2+2M+5}{M^2+M+4}}.$$

The remaining solutions of (31) lead to the same value of $|a_{2(2)}|$.

c) Finally, suppose that $f^* \in \Sigma_{(3)}^M$.

Then, from the equation (5), ($m = 3$), it follows that the coefficient $a_{2(3)}$ of $f^*(z)$ is given by the formula

$$(33)\qquad a_{2(3)} = \frac{1-M}{6M^2}\big[(M^2+M+4)(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)^2 + 2(M^2+M+1)(\beta_1\sigma_1^3+\beta_2\sigma_2^3+\beta_3\sigma_3^3) + {} - 3(M^2+M+2)(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)(\beta_1\sigma_1^2+\beta_2\sigma_2^2+\beta_3\sigma_3^2)\big].$$

Analogously as in b), we verify that the numbers $\sigma_1, \sigma_2, \sigma_3, \beta_1, \beta_2, \beta_3$ associated with $f^*(z)$ satisfy the following system of equations:

$$(34)\qquad \begin{cases} \beta_1+\beta_2+\beta_3 = 2 \\ \sigma_1^2\sigma_2^2\sigma_3^2 = 1 \\ \sigma_1+\sigma_2+\sigma_3 = \dfrac{3}{4}\,\dfrac{M^2+M+2}{M^2+M+1}(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3) \\ (\sigma_1+\sigma_2+\sigma_3)^2+2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3) = \\ = \dfrac{3}{2}\,\dfrac{M^2+M+4}{M^2+M+1}(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)^2 + \\ -\dfrac{3}{2}\,\dfrac{M^2+M+2}{M^2+M+1}(\beta_1\sigma_1^2+\beta_2\sigma_2^2+\beta_3\sigma_3^2). \end{cases}$$

From (34) it follows that

$$\sigma_1 = -1,\quad \sigma_2 = e^{\frac{\pi}{3}i},\quad \sigma_3 = e^{-\frac{\pi}{3}i},\quad \beta_1 = \beta_2 = \beta_3 = \tfrac{2}{3},$$

and, by (33)

$$(33')\qquad a_{2(3)} = \frac{2}{3}M\left(1-\frac{1}{M^3}\right).$$

The remaining solutions of (34) lead to the same value of $|a_{2(3)}|$. Since the inequalities

$$\frac{2}{M}\left(1-\frac{1}{M}\right) < \frac{2}{9M}\left(1-\frac{1}{M}\right)\sqrt{\frac{2}{3}\,\frac{(2M^2+2M+5)^3}{M^2+M+4}} < \frac{2}{3}M\left(1-\frac{1}{M^3}\right)$$

are fulfilled for every $M > 1$, we obtain, using (29), (32), (33'), that

$$a_{2(1)} < a_{2(2)} < a_{2(3)}.$$

Hence it follows that:

For every function $f \in \Sigma^M$ we have

$$(35) \qquad |a_2| \leqslant \frac{2}{3} M\left(1 - \frac{1}{M^3}\right).$$

From (22), (28) and (35) we obtain

Theorem 4. *If a function $f(z)$ belongs to the class Σ^M, then*

$$(36) \qquad |a_n| \leqslant \frac{2}{n+1} M\left(1 - \frac{1}{M^{n+1}}\right) \quad \textit{for } n = 0, 1, 2.$$

The estimation (36) *is sharp and the equality in* (36) *takes place for the function given by the equation*

$$\frac{1}{f(z)}\left(f^{n+1}(z)+1\right)^{\frac{2}{n+1}} = \frac{M}{z}\left(z^{n+1}+1\right)^{\frac{2}{n+1}}.$$

Finally, let us observe that the above results suggest that the estimation (36) in the class Σ^M holds for any natural n. Moreover, it is easy to see, after a suitable normalization of meromorphic quasi-starlike functions ($a_{-1} = 1$), that we can obtain from (36) the analogous results for the class Σ^* obtained earlier by Clunie [2] and Pommerenke [4].

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STRESZCZENIE

Autorka rozpatruje klasę Σ^M funkcji meromorficznych quasi-gwiaździstych, określoną warunkiem (2) i znajduje postać ogólną funkcji ekstremalnych dla pewnych funkcjonałów w tej klasie. Jako zastosowanie znajduje dokładne oszacowania współczynników Laurenta a_n ($n = 0, 1, 2$) w rozważanej klasie.

РЕЗЮМЕ

Автор занимается классом Σ^M мероморфных квази-звездных функций, который определен условием (2) и получает общий вид экстремальных функций для некоторых функционалов в этом классе. В применении дает точную оценку коэффициентов Лорана a_n, $(n = 0, 1, 2)$ в этом классе.