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On the Minorant Sets for Univalent Functions

O zbiorach minoryzacji dla funkcji jednolistnych

Множества минорнзации однолистных функций

1. Let S be the class of functions $f(z) = z + a_2 z^2 + \dots$, regular and univalent in the unit disk K and let S^* be the subclass of functions $f \in S$ starshaped w.r.t. the origin. From the starshapedness property and from the Cauchy-Riemann equations it follows immediately that

$$(1.1) \qquad \frac{1}{r}\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) = \frac{\partial}{\partial r}\log|f(re^{i\theta})| > 0,$$

which means that $|f|$ strictly increases on the segment $[0, a]$. Hence, given $a \in K$, $a \neq 0$, we have: $|f(z)| < |f(a)|$ for any $z \in (0, a)$, and any $f \in S^*$.

Thus, given a fixed subclass $S_0 \subset S^*$ and a fixed $a \in K$, $a \neq 0$, we are led to the determination of the set $\mathscr{D}(a, S_0)$ which is defined as the maximal subset of K such that for any $f \in S_0$ and any $z \in \mathscr{D}(a, S_0)$ we have: $|f(z)| < |f(a)|$. In what follows we shall call $\mathscr{D}(a, S_0)$ the minorant set associated with a and S_0. By our previous remark, $(0, a) \subset \mathscr{D}(a, S_0)$. Obviously,

$$\mathscr{D}(a, S_0) = \bigcap_{f \in S_0} D_f(a),$$

where

$$D_f(a) = \{z \in K : |f(z)| < |f(a)|\}.$$

In this paper we determine $D(a, S_0)$ for the class S^*_a of starshaped functions of order a, in case $a = 0$, $a = 1/2$ ($S^*_0 = S^*$), as well as for the class S^c of convex functions. The class S^*_a is defined by the condition: $\operatorname{re}[zf'(z)/f(z)] > a$, where $0 \leqslant a < 1$.

Suppose that S_0 has the following property:

(1.2) $$\text{if } f \in S_0 \text{ and } |\eta| \leqslant 1, \text{ then } \eta^{-1} f(\eta z) \in S_0.$$

Obviously S^* and S^c have property (1.2). Moreover, for any S_0 with the property (1.2) we have:

1° $$\mathscr{D}(ae^{it}, S_0) = e^{it}\mathscr{D}(a, S_0) \text{ for any real } t,$$

2° $$\text{if } 0 < r < R < 1, \text{ then } \mathscr{D}(r, S_0) \subset \mathscr{D}(R, S_0).$$

In order to determine $\mathscr{D}(a, S_a^*)$, $a = 0, 1/2$, we use the following result [5]:

If $f \in S_a^*$, then for fixed r, $0 < r < 1$, fixed $z \in K$ $(z \neq 0, r)$ and f ranging over S_a^* the set of all values of

$$[rf(z)/zf(r)]^{1/2(1-a)}$$

is the closed disk $\mathscr{K}_{r,z}$ with the boundary

(1.3) $$w(t) = (1 - re^{it})/(1 - ze^{it}),\ t \in [-\pi, \pi].$$

2. We now prove

Theorem 1. *If ϱ, θ are polar coordinates then the boundary of $\mathscr{D}(r, S^*)$ has the equation*

(2.1) $$\varrho = \frac{1}{2r}\left[r^2 + 4r\sin\frac{\theta}{2} + 1 - \sqrt{\left(r^2 + 4r\sin\frac{\theta}{2} + 1\right)^2 - 4r^2}\,\right], \theta \in [0, 2\pi].$$

Proof. Suppose that $f \in S$ and $\zeta \in \mathscr{K}_{r,z}$. Then $f(z)/f(r) = \dfrac{\zeta^2 z}{r}$ is contained in the domain enclosed by the curve

(2.2) $$u(t) = (z/r)[(1 - re^{it})/(1 - ze^{it})]^2,\ t \in [-\pi, \pi].$$

Therefore $z \in \mathscr{D}(r, S^*)$, iff for any $t \in [-\pi, \pi]$ we have: $|(z/r) \times [(1 - re^{it})/(1 - ze^{it})]^2| < 1$, which is equivalent to

(2.3) $$2r\,\mathrm{re}[e^{-it}(z - |z|)] < (r - |z|)(1 - r|z|),\ t \in [-\pi, \pi].$$

It follows from (2.3) that

(2.4) $$\mathscr{D}(r, S^*) = \{z\colon 2r|z - |z|| < (r - |z|)(1 - r|z|)\}.$$

Putting $z = \varrho e^{i\theta}$ in (2.4) we easily obtain (2.1).

In an analogous way we can prove

Theorem 2. *If ϱ, θ are polar coordinates then the boundary of $\mathscr{D}(r, S_{1/2}^*)$ has the equation*

(2.5) $$\theta = \pm\arccos\left[\frac{r^2 + \varrho^2}{2r\varrho} - \frac{1}{2r\varrho}\left(\frac{r^2 - \varrho^2}{2r\varrho}\right)^2\right], \varrho \in \left[\frac{r}{1 + 2r}, r\right].$$

Proof. If is f ranging over $S^*_{1/2}$ and z, r are fixed then the set of all values $f(z)/f(r)$ is a closed disk whose boundary has the equation

$$v(t) = (z/r)(1-re^{it})(1-ze^{it})^{-1},\ t\in[-\pi,\pi], \tag{2.6}$$

which is a consequence of (1.3).

Hence $z\in\mathscr{D}(r, S^*_{1/2})$, iff for any $t\in[-\pi,\pi]$ we have: $|(z/r)(1-re^{it})\times\times(1-ze^{it})^{-1}|<1$, which is equivalent to

$$2r\,\mathrm{re}\{e^{-it}(rz-|z|^2)\}<r^2-|z|^2,\ t\in[-\pi,\pi]. \tag{2.7}$$

Now, (2.7) implies

$$\mathscr{D}(r, S^*_{1/2}) = \{z\colon 2r|rz-|z|^2|<r^2-|z|^2\}. \tag{2.8}$$

Putting $z=\varrho e^{i\theta}$ in (2.8) we obtain the equation of the boundary of $\mathscr{D}(r, S^*_{1/2})$:

$$2r\varrho\sqrt{r^2+\varrho^2-2r\varrho\cos\theta} = r^2-\varrho^2. \tag{2.9}$$

It is easily verified that $\dfrac{d\varrho}{d\theta}<0$ for $\theta\in(0,\pi)$, hence we can obtain θ as a function $\varrho\in[r/(1+2r), r]$ in the form (2.5).

Corollary.

$$\mathscr{D}(r, S^*_{1/2}) = \mathscr{D}(r, S^c). \tag{2.10}$$

Proof. Obviously $\mathscr{D}(r, S^*_{1/2})\subset\mathscr{D}(r, S^c)$ since $S^c\subset S^*_{1/2}$. On the other hand, for the family of functions $\{\eta^{-1}f_0(\eta z)\}$, where $|\eta|=1$ and $f_0(z) = z(1+z)^{-1}$, the inequality $|f(z)|<|f(r)|$ leads to the set on the right-hand side of the equality (2.8). This proves the Corollary.

3. We now give an example showing that the determination of the set $\mathscr{D}(r, S_0)$ provides the solution of some problems in the theory of subordination.

Let us start with some definitions and notations.

Suppose that A is the class of functions $f(z) = a_1z+a_2z^2+\ldots$, $a_1\geqslant 0$, analytic in the unit disk K and let B be the subclass of A consisting of all $\omega\in A$ with $|\omega(z)|<1$, $z\in K$.

Suppose, moreover, that $H(z)$ denotes for a fixed $z\in K$ the closed convex circular triangle whose boundary consists of an arc of the circle $\{\zeta\colon|\zeta| = |z|^2\}$ and of two circular arcs through z tangent to the former circle. The region $H(z)$ has following properties:

(i) $\quad H(ze^{it}) = e^{it}H(z)$ for any real t,

(ii) $\quad$ if $|z|<|\zeta|$, $\arg z = \arg\zeta$, then $H(z)\subset H(\zeta)$.

Moreover, if R, t are polar coordinates then the boundary points of $H(r)$ satisfy the equations

(3.1) $$R^2+R(1-r^2)\sin t-r^2 = 0,\ t\in\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3}{2}\pi, 2\pi\right],$$

(3.2) $$R = r^2, \quad t\in\left[\frac{\pi}{2},\frac{3}{2}\pi\right].$$

It is easy to verify (cf. [3], p. 327) that the set $H(z)$ defined above is identical with the set $\{W\colon W = \omega(z), \omega\in B\}$.

The function $f\in A$ is said to be subordinate to $F\in S_0$ in K (which is denoted: $f \dashv F$) if $f = F\circ\omega$ with $\omega\in B$.

M. Biernacki was the first [1] who considered the following problem: Given the classes A, S_0 evaluate the number

(3.31) $$r_0 = r(A, S_0) = \inf r(f, F),$$

where

(3.32) $$r(f, F) = \sup\{r\colon [(f\not\equiv F)\wedge(0<|z|<r)] \Rightarrow |f(z)|<|F(z)|\},$$

for a fixed pair (f, F) such that $f\in A, F\in S_0, f\dashv F$, the infimum in (3.31) being taken with respect to all such pairs.

In the above given notation we can state a general method of evaluating $r(A, S_0)$ in terms of the set $\mathscr{D}(a, S_0)$. We have

Theorem 3. Suppose that a fixed subclass S_0 of normalized, univalent functions has the property (1.2). Then

(3.4) $$r(A, S_0) = \sup\{r\colon H(r)\subset\mathscr{D}(r, S_0)\}.$$

The proof can be easily derived from the properties of the above introduced sets $H(r)$, $\mathscr{D}(r, S_0)$ and is omitted here.

We shall apply Theorem 3 to the evaluation of $r(A, S^*)$ and $r(A, S^c)$. An alternate method was applied earlier in [4] and [2] in evaluating these constants. We have

Theorem 4.

(3.5) $$r(A, S^*) = \tfrac{1}{2}(3-\sqrt{5}),$$

(3.6) $$r(A, S^c) = \tfrac{1}{3}.$$

Proof. We first verify (3.5). The boundary of $H(r)$ has according to (3.1), (3.2) the following equation

(3.7) $$R(t) = \begin{cases} \tfrac{1}{2}\left[-(1-r^2)\sin t+\sqrt{(1-r^2)^2\sin^2 t+4r^2}\right], & t\in\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3}{2}\pi, 2\pi\right] \\ r^2, & t\in\left[\frac{\pi}{2},\frac{3}{2}\pi\right]. \end{cases}$$

If $H(r) \subset \mathcal{D}(r, S^*)$ then $R(\pi) < \varrho(\pi)$ and hence $0 < r < \frac{1}{2}(3-\sqrt{5})$ by (2.1). We have to show that

$$(3.8)\qquad R(t) < \varrho(t), \text{ for } t\in(0,\pi) \text{ and } r = \tfrac{1}{2}(3-\sqrt{5}).$$

For a fixed r the function $\varrho(t)$ decreases in $(0, \pi)$ and therefore (3.8) holds obviously for $t\in\left[\frac{\pi}{2}, \pi\right)$. On the other hand, for $t\in\left(0, \frac{\pi}{2}\right)$ the inequality (3.8) can be written in the following form:

$$(3.9)\qquad \sqrt{5\sin^2 t+4}-\sqrt{5}\sin t < \frac{1}{r}\left[3+4\sin\frac{t}{2}-\sqrt{\left(3+4\sin\frac{t}{2}\right)^2-4}\right].$$

Leaving the functions containing square roots on the left hand side and then squaring both sides twice we bring the inequality (3.9) to the form

$$(3.10)\qquad 12+8\sin\frac{t}{2}-15\cos\frac{t}{2}-5\sin t\cos\frac{t}{2} < 10\sin t,\ t\in\left[0, \frac{\pi}{2}\right)$$

in view of the equality $1-r^2 = \sqrt{5}r$ with $r = \frac{1}{2}(3-\sqrt{5})$. If the left hand side and the right hand side in (3.10) are denoted $g(t)$ and $h(t)$, resp., then $g''(t) > 0$, $h''(t) < 0$ for $t\in\left(0, \frac{\pi}{2}\right)$, $g(0) < h(0)$, $g\left(\frac{\pi}{2}\right) < h\left(\frac{\pi}{2}\right)$. This proves (3.10), as well as (3.8) for $t\in\left(0, \frac{\pi}{2}\right)$ and the equality (3.5) follows.

We now prove (3.6). Again $R(\pi) < \varrho(\pi)$ and in view of (2.9) we obtain $0 < r < \frac{1}{2}$. We now prove that

$$(3.11)\qquad R(t) < \varrho(t) \text{ for } t\in(0,\pi) \text{ and } r = \tfrac{1}{2}.$$

If $t\in\left(\frac{\pi}{2}, \pi\right)$, then (3.11) is obviously true and if $t\in\left(0, \frac{\pi}{2}\right)$, then (3.11) takes the form

(3.12)

$$\arcsin\frac{1-4\varrho^2}{3\varrho} < \arccos\left[\frac{1+4\varrho^2}{4\varrho}-\frac{1}{\varrho}\left(\frac{1-4\varrho^2}{4\varrho}\right)^2\right],\ \varrho\in\left(\frac{1}{2}\sqrt{\frac{5-\sqrt{17}}{2}}, \frac{1}{2}\right).$$

The left hand side in (3.12) is obtained from (3.1) by expressing t as a function of ϱ, whereas the right hand side of (3.12) is obtained from (2.5) by putting $r = \frac{1}{2}$. After some obvious transformations (3.12) takes the form

$$(3.13)\qquad 3(12\varrho^2-1) < 16\varrho^2\sqrt{9\varrho^2-(1-4\varrho^2)^2},$$

and this is easily verified by elementary calculus. This proves Theorem 4.

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STRESZCZENIE

Niech S_0 oznacza ustaloną podklasę klasy S funkcji $f(z) = z + a_2z^2 + \ldots$, analitycznych i jednolistnych w kole jednostkowym K. W pracy tej zajmujemy się problemem wyznaczenia zbioru

$$\mathscr{D}(a, S_0) = \bigcap_{f \in S_0} D_f(a),$$

gdzie

$$D_f(a) = \{z \in K: |f(z)| < |f(a)|\}, a \in K,$$

w przypadku, gdy S_0 jest klasą funkcji gwiaździstych lub wypukłych (twierdzenia 1 i 2). Używając zbioru $\mathscr{D}(a, S_0)$ podajemy w twierdzeniu 3 ogólną metodę rozwiązywania problemu M. Biernackiego dla funkcji podporządkowanych i w oparciu o nią powtarzamy niektóre wcześniej i na innej drodze uzyskane rezultaty (twierdzenie 4).

РЕЗЮМЕ

Пусть S_0 обозначает фиксированный подкласс класса S функций $f(z) = z + a_2z^2 + \ldots$, аналитических и однолистных в единичном круге K. В этой работе изучается проблема определения множества

$$D(a, S_0) = \bigcap_{f \in S_0} D_f(a)$$

где $D_f(a) = \{z \in K: |f(z)| < |f(a)|\}$, $a \in K$, если $S_0 = S^*$, S^c (теоремы 1, 2).

Теорема 3 содержит общий метод решения проблемы М. Бернацкого для подчиненных функций. По этому методу получены некоторые результаты данные ранее Г. М. Голузиным и другими авторами (теорема 4).