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On a Theorem of M. Biernacki Concerning Subordinate Functions

O twierdzeniu M. Biernackiego dotyczącym funkcji podporządkowanych

К теореме М. Бернацкого, относящейся к подчиненным функциям

1. Introduction

Let the functions f and F be regular in the unit disk K_1. The function f is said to be subordinate to F in K_1 if there exists a function ω regular in K_1 such that $\omega(0) = 0$, $|\omega(z)| < 1$ in K_1 and $f = F \circ \omega$ in K_1.

In this case we write $f \dashv F$. M. Biernacki [4] was first to consider the following problem. Suppose S_0 is a fixed subclass of the class S of functions regular and univalent in the unit disk K_1 and subject to the usual normalization. Suppose, moreover, that $f \dashv F$, $F \in S_0$ and f satisfies some additional conditions (e.g. $f(z)/f'(0) \in S_0$). Find the greatest number $r_0 \in (0, 1)$ such that the above stated conditions imply $|f(z)| \leqslant |F(z)|$ for any $z \in K_{r_0} = \{z: |z| < r_0\}$ and any pair of admissible functions f, F. Variant forms of this problem were investigated by Biernacki [5], Golusin [6], Shah Tao — shing [10], Bielecki and Lewandowski [2], [3]. In [1] Bielecki and Lewandowski have given a general method of evaluating r_0 in case we know the estimate of $\arg(zF'(z)/F(z))$ for $F \in S_0$ and $f(z) = a_1 z + a_2 z^2 + \ldots$ satisfies the following conditions: $f'(0) > 0, f(z) \neq 0$ for $z \neq 0$. In this paper we give a different method of evaluating the constant r_0 for $F \in S_0$, where S_0 is a fixed subclass of S and $f(z) = a_1 z + a_2 z^2 + \ldots$ is regular in K_1. This method can be effectively applied in case we know the region of variability of the expression $F(z_2)/F(z_1)$, where z_1, z_2 are arbitrary fixed points of K_1 and F ranges over S_0.

2. Auxiliary lemmas

In what follows we shall need Lemma 1 which is a generalization of a result of Rogosinski.

Lemma 1. *Let B_n be the class of functions ω such that $\omega(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$, is regular in K_1, $a_n \geqslant 0$, $|\omega(z)| < 1$ in K_1. Suppose z_1 is a fixed point in K_1. Let $H_n(z_1)$ be the closed domain whose boundary is the union of the semicircle $z = i|z_1|z_1^n \cdot e^{it}$, $0 \leqslant t \leqslant \pi$, and two circular arcs l_1, l_2 joining z_1^n to $i|z_1|z_1^n$ and $-i|z_1|z_1^n$ to z_1^n resp., both arcs l_1, l_2 being tangent to the circle $|z| = |z_1|^{n+1}$. Then for each $\omega \in B_n$ we have $\omega(z_1) \in H_n(z_1)$ and, conversely if $\zeta_1 \in H_n(z_1)$, then there exists $\omega \in B_n$ such that $\omega(z_1) = \zeta_1$. The arcs l_1, l_2 have the following parametric representation:*

$$l_1\colon\ w = z_1^n(a + i|z_1|)(1 + ia|z_1|)^{-1},\quad 0 \leqslant a \leqslant 1. \tag{2.1}$$

$$l_2\colon\ w = z_1^n(a - i|z_1|)(1 - ia|z_1|)^{-1},\quad 0 \leqslant a \leqslant 1. \tag{2.2}$$

The proof of this lemma can be easily obtained in an analogous way as in [7], or [8]. In what follows we write $H(z)$ instead of $H_1(z)$.

Lemma 2. *Suppose S_0 is a fixed closed subclass of S such that*

$$F \in S_0 \quad \text{and} \quad |\eta| \leqslant 1 \quad \text{implies} \quad \eta^{-1}F(\eta z) \in S_0. \tag{2.3}$$

Suppose $Q_n(z_1, S_0) = \{u\colon u = F(z_2)/F(z_1)\}$, where z_1 is a fixed point of K_1, z_2 ranges over $H_n(z_1)$ and F ranges over S_0. Then $Q(z_1, S_0) = Q_1(z_1, S_0)$ has the following properties:

(i) $Q(z_1, S_0)$ *is connected,*

(ii) $Q(z_1, S_0) = Q(\eta z_1, S_0)$, *where $\eta z_1 = |z_1|$,*

(iii) $0 < r < R < 1$, *then $Q(r, S_0) \subset Q(R, S_0)$.*

Proof. The property (i) is obvious. By (2.3) we see that any $F \in S$ can be continuously deformed into identity. Hence we easily deduce that $Q(z_1, S_0)$ is arcwise connected.

(ii). Take arbitrary $z_1 \in K_1$. For any $u \in Q(z_1, S_0)$ we can find $f \in S_0$ and $z_2 \in H(z_1)$ such that $u = f(z_2)/f(z_1)$. Take now an arbitrary η such that $|\eta| = 1$. By (2.3), $F(z) = \eta^{-1}f(\eta z) \in S_0$. Obviously $\zeta_2 = \eta z_2 \in H(\eta z_1)$. Hence $u = F(\zeta_2)/F(\eta z_1) \in Q(\eta z_1, S_0)$. This means $Q(z_1, S_0) \subset Q(\eta z_1, S_0)$ and consequently $Q(z_1, S_0) = Q(\eta z_1, S_0)$.

(iii) Suppose $0 < r < R < 1$ and $r = \lambda R$ $(0 < \lambda < 1)$. If $u \in Q(r, S_0)$ then we can find $f \in S_0$ and $z_2 \in H(r)$ such that $u = f(z_2)/f(r)$. If $F(z) = \lambda^{-1}f(\lambda z)$, then $F \in S_0$ by our assumption. Moreover, $u = f(z_2)/f(r) = F(\zeta_2)/F(R)$ where $\lambda\zeta_2 = z_2$. From the definition of $H(r)$ it follows that $z_2 \in H(r)$ implies $\lambda^{-1}z_2 = \zeta_2 \in H(R)$ and consequently $u \in Q(R, S_0)$. Hence (iii) follows.

3. Main result

Let A_n be the class functions f analytic in the unit disk K_1 and such that $f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$, where $n \geqslant 1$ and $a_n \geqslant 0$. Put $\mathscr{C}K_1 = \{z\colon |z| \geqslant 1\}$. The main result of this paper is the following:

Theorem. *Consider a fixed subclass S_0 of the class S satisfying the conditions of Lemma 2. Put*

$$r_0 = \sup\{r\colon [Q_n(z_1, S_0) \cap \mathscr{C}K_1] - \{1\} = \emptyset; |z_1| = r\}. \tag{3.1}$$

Then the conditions: $f \in A_n$, $F \in S_0$, $f \dashv F$ in K_1, $f \neq F$ imply $|f(z)| < |F(z)|$ if and only if $0 < |z| < r_0$.

Proof. We first prove the sufficiency. Suppose there exist two functions $f \in A_n$, $F \in S_0$, $f \neq F$, such that $f \dashv F$ in K_1 and $|f(z_1)| \geqslant |F(z_1)|$ with some $0 < |z_1| < r_0$. Then $f(z) = F(\omega(z))$ with $f \in A_n$ means that $\omega \in B_n$. We have $f(z_1) = F(\omega(z_1)) = F(z_2)$, where $z_2 = \omega(z_1)$, i.e. $z_2 \in H_n(z_1)$, $z_2 \neq z_1$. Now, $|F(z_2)| \geqslant |F(z_1)|$ means that $u = F(z_2)/F(z_1)$ has absolute value $\geqslant 1$ without being equal 1. This implies that $[Q_n(z_1, S_0) \cap \mathscr{C}K_1] - \{1\}$ is not empty. Now, $0 < |z_1| < r_0$, and this contradicts the definition of r_0.

Suppose now $z_1 \in K_1$ is such that $|z_1| > r_0$. This means that there exists $u \neq 1$ and $F \in S_0$ such that $|u| \geqslant 1$, $u = F(z_2)/F(z_1)$ and $z_2 \in H_n(z_1)$. By Lemma 1 we can find $\omega \in B_n$ such that $z_2 = \omega(z_1)$. Consider now $f = F \circ \omega$. Obviously $f \in A_n$. Moreover, $f(z_1) = F(\omega(z_1)) = F(z_2) = uF(z_1)$ and this implies $|f(z_1)| \geqslant |F(z_1)|$. This means that $|f(z)| < |F(z)|$ not necessarily holds for $|z_1| > r_0$. Suppose now $\{z_k\}$ is an arbitrary sequence of complex numbers such that $r_0 < |z_k| < 1$, $\{|z_k|\}$ strictly decreases and $\lim_{k\to\infty} |z_k| = r_0$. As above we can find $F_k \in S_0$ and $\omega_k \in B_n$ such that for $f_k = F_k \circ \omega_k$ we have $|f_k(z_k)| \geqslant |F_k(z_k)|$. Since S_0 and B_n are compact families, we can find by choosing suitable subsequences a point z_0 with $|z_0| = r_0$, and two functions $F_0 \in S_0$, $\omega_0 \in B_n$, such that $|f_0(z_0)| \geqslant |F_0(z_0)|$ where $f_0 = F_0 \circ \omega_0$. Thus $|f(z)| < |F(z)|$ not necessarily holds for $|z_0| = r_0$.

4. An application

We now apply Theorem 1 with $S_0 = S^*$ which is the subclass of functions starlike w.r.t. the origin to obtain in a different way a well known result due to G. M. Golusin [6]. Suppose $n = 1$. If $F \in S^*$ and $z_1, z_2 \in K_1$ ($z_1 \neq 0$), then the point $[F(z_2)/F(z_1)]^{1/2}$ is situated in the closed disk with boundary

$$w = (z_2/z_1)^{1/2}(1 - z_1 e^{it})(1 - z_2 e^{it})^{-1}, \quad -\pi \leqslant t \leqslant \pi, \tag{4.1}$$

whose centre and radius are

$$s = (1 - |z_2|^2)^{-1}(q - |z_2|^2 q^{-1}), \quad R = |z_2|(1 - |z_2|^2)^{-1}|q - q^{-1}|,$$

where $q = (z_2/z_1)^{1/2}$, and conversely, if u is inside the disk (4.1), there exists $F \in S^*$ such that $u = [F(z_2)/F(z_1)]^{\frac{1}{2}}$, cf. e.g. [9]. We take here those branches of square root which give 1 as z_2 approaches z_1. In view of Lemma 2 we can assume that $z_1 = r\in(0,1)$ and $z_2\in H(r)$. Since w^2 as given by (4.1) is an analytic function of $z_2\in H(r)$ its modulus attains a maximum on $\partial H(r)$. In order to prove that $[Q(r, S^*)\cap \mathscr{C}K_1]-\{1\}$ is empty for some $r\in(0,1)$ we have only to show that

$$|(z/r)[(1-re^{it})/(1-ze^{it})]^2| < 1 \tag{4.2}$$

for any $z\in H(r)-\{r\}$. The boundary $\partial H(r)$ of $H(r)$ is given by the following equations:

$$z = z_0(\theta) = r^2 e^{i\theta}, \qquad \frac{\pi}{2}\leqslant\theta\leqslant\frac{3\pi}{2} \tag{4.3}$$

$$z = z_1(a) = r(a+ir)/(1+iar), \qquad 0\leqslant a\leqslant 1; \tag{4.41}$$

$$z = z_2(a) = r(a-ir)/(1-iar), \qquad 0\leqslant a\leqslant 1. \tag{4.42}$$

Suppose z is situated on the arc (4.3). Then (4.1) gives

$$|w|^2 = |w(t,\theta)|^2 = r|(1-re^{ti})/(1-r^2e^{i(t+\theta)})|^2;$$

hence $\max\limits_{t,\theta}|w(t,\theta)|^2 = |w(-\pi,\pi)|^2 = r(1-r)^{-2}$. Now, if $0 < r < \frac{1}{2}(3-\sqrt{5})$, then $r(1-r)^{-2} < 1$ and (4.2) holds for z given by (4.3). Suppose now z is represented by (4.41), or (4.42). From (4.1) we obtain

$$|w|\leqslant|s|+R = (1-r^4)^{-1}\sqrt[4]{(a^2+r^2)(1+a^2r^2)}\times \sqrt{(1-ar^2)^2+r^2(a-r^2)^2}+r(1-a)\sqrt{1+r^2}]. \tag{4.5}$$

Now, by an elementary calculation we show that

$$(1-r^4)^{-2}\sqrt{(a^2+r^2)(1+a^2r^2)}\times \times[\sqrt{(1-ar^2)^2+r^2(a-r^2)^2}+r(1-a)\sqrt{1+r^2}]^2 < 1, \tag{4.6}$$

holds for all $a\in(0,1)$ and $r\in(0,\sqrt{2}-1)$. This implies that (4.5) holds on both open arcs (4.41), (4.42) with $r_1 = \frac{1}{2}(3-\sqrt{5})\in(0,\sqrt{2}-1)$. Hence (4.2) holds with $r < r_1$. This shows that the sets $[Q(r, S^*)\cap\mathscr{C}K_1]-\{1\}$ are empty for $r\in(0,r_1)$. On the other hand, the set $[Q(r_1, S^*)\cap\mathscr{C}K_1]-\{1\}$ contains the point -1. In fact $z_2 = -r_1^2\in H(r_1)$ and for $F(z) = z(1+z)^{-2}$ we have $F(z_2)/F(z_1) = -r_1(1-r_1)^{-2} = -1$. This proves that r_0 as given by (3.1) with $S_0 = S^*$ is equal to $r_1 = \frac{1}{2}(3-\sqrt{5})$.

Suppose now $n\geqslant 2$. The boundary $\partial H_n(z_1)$ of $H_n(z_1)$ is given by the equations (2.1), (2.2) and the equation

$$z = z(\theta) = r^{n+1}e^{i\theta},\ \arg z_1^n+\frac{\pi}{2}\leqslant\theta\leqslant\arg z_1^n+\frac{3\pi}{2}. \tag{4.7}$$

In an analogous manner as above we obtain with (4.7) and (4.1), that $\max_{\theta,t}|w^2(t,\theta)| = \left|w^2\left(-\frac{1}{n}\pi, \frac{n+1}{n}\pi\right)\right| = r^n\left(\frac{1+r}{1-r^{n+1}}\right)^2$. With (2.1) or (2.2) and (4.1) we have $\max_{a,t,\theta}|w^2(a,t,\theta)| = \max_{a,t,\theta} r^{n-1}\left|\frac{a+ir}{1+iar}\left[(1-re^{i(t+\theta)}) \times \right.\right.$ $\left.\left.\times\left(1-\frac{a+ir}{1+iar}r^n e^{i(nt+\theta)}\right)^{-1}\right]^2\right| = \left|w^2\left(1, -\frac{1}{n-1}\pi, \frac{n}{n-1}\pi\right)\right| = r^{n-1}\left(\frac{1+r}{1-r^n}\right)^2$.
Hence, the following conditions: $n \geqslant 2, f\in A_n, F\in S^*, f \dashv F$ and $0 < |z| < r_0(n)$ imply $|f| < |F|$, where $r_0(n)$ is the least positive root of the equation $r^{n-1}\left(\frac{1+r}{1-r^n}\right)^2 = 1$

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REFERENCES

[1] Bielecki, A. et Lewandowski, Z. —, *Sur une generalization de quelques theoremes de M. Biernacki sur les fonctions analytiques*, Ann. Polon. Math. 12 (1962), 65—70.

[2] —, *Sur certaines majorantes des fonctions holomorphes dans cercle unite*, Coll. Math. 9 (1962) 299-303.

[3] —, *Sur un type de fonctions holomorphes subordonnees*, Folia Societatis Scientiarum Lublinensis 2 (1962), 92-94.

[4] Biernacki, M. —, *Sur quelques majorantes de la theorie des fonctions univalentes*, C. R. Acad. Sci. Paris 201 (1935), 256-258.

[5] —, *Sur les fonctions univalentes*, Mathematica 12 (1936), 49-64.

[6] Голузин, Г. М. —, Мажорация подчиненных аналитических функций I, Математический сборник, 29 (1951), 209—224.

[7] —, Геометрическая теория функций комплексного переменного, Москва—Ленинград 1952.

[8] Lewandowski, Z. —, *Starlike Majorants and Subordination*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, 15 (1961), 79-84.

[9] —, *Sur certaines classes de fonctions univalentes dans le cercle — unite*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, 13 (1959), 115-126.

[10] Shah Tao-shing, *Goluzin's Number* $\frac{1}{2}(3-\sqrt{5})$ *Is the Radius of Superiority in Subordination*, Sci. Rec. 1 (1957), 258-261.

Streszczenie

Niech A_n będzie klasą funkcji $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $a_n \geqslant 0$, regularnych dla $|z| < 1$, zaś S_0 niech oznacza pewną ustaloną podklasę funkcji F, klasy S.

W pracy tej dowodzę, że nierówność $|f(z)| < |F(z)|$, $0 < |z| < r_0$, przy założeniu $f \dashv_1 F$, zachodzi wtedy i tylko wtedy, gdy liczba r_0 jest określona przez warunek (3.1).

Opierając się na powyższym twierdzeniu wyznaczam stałą r_0 znalezioną wcześniej przez G. M. Gołuzina, w przypadku gdy $n = 1$ i $S_0 = S^*$.

Резюме

Пусть A_n обозначает класс функций $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $a_n \geqslant 0$, голоморфных в круге $|z| < 1$, а S_0 — подкласс функций F класса S.

В работе автор доказывает, что неравенство $|f(z)| < |F(z)|$, $0 < |z| < r_0$ при условии $f \dashv F$ выполняется тогда и только тогда, если число r_0 определено условием (3.1).

Опираясь на доказанную теорему, автор определяет константу r_0 (найденую ранее Г. М. Голузиным) при $n = 1$ и $S_0 = S^*$.