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A Note on Qualitative Conditions for the Strong Law of Large Numbers

Uwaga o jakościowych warunkach dla mocnego prawa wielkich liczb

Замечание о качественных условиях для усиленного закона больших чисел

E. Franckx in [1] considers a sequence of uniformly bounded random variables $\{X_n\}$ and formulates his qualitative criterion: a necessary and sufficient condition for the strong law of large numbers (S. L. L. N) for such a sequence is the existence of a characteristic subsequence $\{S_{n_k}/n_k\}$ of $\{S_n/n\}$, $S_n = \sum_{j=1}^{n} X_j$ such that

(1) $$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1,$$

(2) $$\frac{S_{n_k} - ES_{n_k}}{n_k} \xrightarrow{\text{a.s.}} 0, \text{ where } S_{n_k} = \sum_{j=1}^{n_k} X_j.$$

In this note a case is considered when random variables are bounded but from one side, i.e. either $X_n \leqslant l$ or $X_n \geqslant l$, where l is a finite number. Without loss on generality either case can be reduced to the case of non negative random variables ($X_n \geqslant 0$), for if l is negative and is a lower bound, we can by adding $-l$ to random variables bring them to $X_n \geqslant 0$, and if l is an upper bound we can by substracting l from random variables and multiplying by -1 bring them to $X_n \geqslant 0$.

Theorem 1. *The S. L. L. N. holds for a sequence of non negative random variables $\{X_n\}$($X_n \geqslant 0$ for all n) with bounded expectations $EX_n \leqslant L$, if and only if there exists a characteristic subsequence (in Franckx's sense).*

Proof. The necessity of condition is obvious from the fact that once a sequence $\{S_n/n - ES_n/n\}$ converges to zero a. s., any subsequence of this sequence converges to zero a.s.

Condition is sufficient.

Since $X_n \geqslant 0$, it follows that for $n_k < n \leqslant n_{k+1}$

$$S_{n_k} - ES_{n_{k+1}} \leqslant S_n - ES_n \leqslant S_{n_{k+1}} - ES_{n_k}. \tag{3}$$

From (2) we have: with probability 1 for sufficiently large k and an arbitrary positive constant ε

$$-\varepsilon n_k + ES_{n_k} < S_{n_k} < ES_{n_k} + \varepsilon n_k$$

and also

$$-\varepsilon n_{k+1} + ES_{n_{k+1}} < S_{n_{k+1}} < ES_{n_{k+1}} + \varepsilon n_{k+1}.$$

Hence using (3) we obtain: with probability 1 for sufficiently large k

$$-\varepsilon n_{k+1} - (ES_{n_{k+1}} - ES_{n_k}) < S_n - ES_n < (ES_{n_{k+1}} - ES_{n_k}) + \varepsilon n_{k+1},$$

i.e.

$$\frac{|S_n - ES_n|}{n} < \frac{ES_{n_{k+1}} - ES_{n_k}}{n_k} + \frac{n_{k+1}}{n_k}\varepsilon, \tag{3'}$$

and since $EX_n \leqslant L$,

$$\frac{|S_n - ES_n|}{n} < \frac{n_{k+1}}{n_k} \cdot \frac{(n_{k+1} - n_k)L}{n_{k+1}} + \frac{n_{k+1}}{n_k} \cdot \varepsilon \to \varepsilon,$$

when $k \to \infty$.

Thus we have: with probability 1 for sufficiently large n

$$\frac{|S_n - ES_n|}{n} < \varepsilon + \eta,$$

where ε and η are arbitrary positive constants, i.e.

$$\frac{S_n - ES_n}{n} \xrightarrow{\text{a.s.}} 0.$$

Corollary. *For a sequence of non negative random variables $\{X_n\}$ with common finite expectation $EX_n = \mu$ the S. L. L. N. holds if and only if there is a characteristic subsequence (in Franckx's sense).* Proof is immediate.

Theorem 2. *The S. L. L. N. holds for a sequence of non negative random variables if there exists such a subsequence of natural numbers $n_k \uparrow \infty$ with $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = c < \infty$ that*

$$\frac{S_{n_k}}{n_k} - E\frac{S_{n_k}}{n_k} \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad \frac{ES_{n_{k+1}} - ES_{n_k}}{n_k} \to 0.$$

The proof follows directly from (3′) in the Theorem 1.

REFERENCES

[1] Franckx, E., *La loi forte des grand nombres des variables uniformement bornées*, Trab. de Estadist., **9** (1958), p. 111-115.

Streszczenie

W pracy tej dowodzi się, że warunkiem koniecznym i dostatecznym na to, aby ciąg zmiennych losowych nieujemnych o ograniczonych wartościach oczekiwanych spełniał mocne prawo wielkich liczb, jest kryterium jakościowe Franckxa. Podaje się również jakościowe warunki dostateczne na to, aby ciąg zmiennych losowych nieujemnych spełniał mocne prawo wielkich liczb.

Резюме

В статье доказывается, что необходимым и достаточным условием для исполнения усиленного закона больших чисел в случае неотрицательных случайных величин, имеющих ограниченные математические ожидания есть качественный критерий Франкса. Кроме того, даются достаточные качественные условия исполнения усиленного закона больших чисел в общем случае неотрицательных случайных величин.