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On the Region of Variability of the Ratio $f(z_1)/f(z_2)$ within the Class S of Univalent Functions

O obszarze zmienności stosunku $f(z_1)/f(z_2)$ w klasie S funkcji jednolistnych

Об области всех возможных значений $f(z_1)/f(z_2)$ в классе S однолистных функций

1. Notations. Statement of results

Let S be the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in the unit disc $K = \{z: |z| < 1\}$.

The determination of the region $D(z_1, z_2)$ of variability of the ratio $f(z_1)/f(z_2)$, where z_1, z_2 are fixed points of K different from 0 and from each other and f ranges over S, is intimately connected with some other unsolved problems in the theory of functions, e.g. with the evaluation of precise bounds of $\arg F(z)$, where $F(z)$ is a univalent function with Montel's normalization: $F(0) = 0$, $F(z_1) = 1$.

Let $\lambda(\zeta) = k^2(\zeta)$ be the elliptic modular function (the Jacobian modulus) defined by the equation $\zeta = iK(1-\lambda)/K(\lambda)$, where $K(\lambda) = \int_0^1 [(1-t^2)(1-\lambda t^2)]^{-1/2} dt$ is real and positive for $0 < \lambda < 1$, cf. [1], [5].

Let $\varphi(z)$ be an arbitrary branch of $[z(z-z_1)(z-z_2)\times(1-\bar{z}_1 z)\times \times(1-\bar{z}_2 z)]^{-1/2}$, defined inside the triangle $[z_0\, z_1\, z_2]$, $z_0 = 0$, and put

$$G_k = \int_0^{z_k} \varphi(\zeta)\,d\zeta, \qquad H_k = \int_0^{z_k} \zeta\varphi(\zeta)\,d\zeta, \qquad k = 1, 2, \tag{1.1}$$

where the integrals are taken along the sides of the triangle.

In this paper we show that all the boundary points of $D(z_1, z_2)$ are situated on the analytic curve $\Gamma(z_1, z_2)$ which is the map under $\lambda(\zeta)$ of the circumference $\gamma(z_1, z_2)$:

$$\zeta = \zeta(\alpha) = 1 \pm \frac{e^{i\alpha}G_2 - H_2}{e^{i\alpha}G_1 - H_1}, \qquad 0 \leqslant \alpha \leqslant 2\pi; \tag{1.2}$$

the sign in (1.2) has to be chosen so that $\gamma(z_1, z_2)$ should lie in the upper half-plane which is possible since the imaginary part of $(e^{i\alpha}G_2-H_2)/(e^{i\alpha}G_1-H_1)$ never vanishes.

The univalent functions corresponding to the points $\lambda(\zeta(\alpha))$ of $\Gamma(z_1, z_2)$ have the form

$$(1.3)\qquad f(z, \alpha) = C\left\{\mathfrak{p}\left[\int_0^z e^{-i\alpha/2}(\zeta-e^{i\alpha})\varphi(\zeta)d\zeta+\frac{1}{2}\Omega_1(\alpha)+\frac{1}{2}\Omega_2(\alpha)\right]+ +e_1(\alpha)+e_2(\alpha)\right\},$$

where C is a constant,

$$(1.4)\qquad e_k(\alpha) = \mathfrak{p}\left(\frac{1}{2}\Omega_k(\alpha)\right), \quad k = 1, 2,$$

and $\mathfrak{p}$ has primitive periods $\Omega_k(\alpha)$, $k = 1, 2$, equal to those of the hyperelliptic integral $\int e^{-i\alpha/2}(\zeta-e^{i\alpha})\varphi(\zeta)d\zeta$ for paths situated inside K.

2. The differential equation of extremal functions

In order to obtain the differential equation of functions corresponding to the boundary $\partial D(z_1, z_2)$ of $D(z_1, z_2)$, we apply Schiffer's variational method, cf. [2], p. 103, and the Lagrange multipliers.

Let $f\in S$ and put $F(z,\varrho) = f(z)+\varrho e^{i\varphi}[f(z)]^2[f(z)-f(u)]^{-1}$, where $\varrho > 0$, φ is real and $|u| < r < 1$. If ϱ is small enough, $F(z,\varrho)$ maps the annulus $r < |z| < 1$ conformally on a doubly connected domain which arises by removing from a simply connected domain B_ϱ the interior of an analytic Jordan curve being the map of $|z| = r$ under $F(z,\varrho)$. The function $f^*(z)$ realizing the conformal mapping of K on B_ϱ so that $f^*(0) = 0$, has the following form

$$(2.1)\qquad f^*(z) = f(z)\left\{1+\varrho e^{i\varphi}\left[\frac{f(z)}{f(z)-f(u)}+\frac{zf'(z)}{f(z)}\left(\frac{f(u)}{uf'(u)}\right)^2\frac{u}{u-z}\right]+ +\varrho e^{-i\varphi}\left[\frac{zf'(z)}{f(z)}\overline{\left(\frac{f(u)}{uf'(u)}\right)^2}\frac{z\bar{u}}{1-\bar{u}z}\right]\right\}+O(\varrho^2)$$

where the term $O(\varrho^2)$ has a uniform estimation on compact subsets of K. We have, moreover,

$$(2.2)\qquad \delta\log\frac{f(z_1)}{f(z_2)} = \log\frac{f^*(z_1)}{f^*(z_2)}-\log\frac{f(z_1)}{f(z_2)}+O(\varrho^2) = \log\frac{f^*(z_1)}{f(z_1)}-\log\frac{f^*(z_2)}{f(z_2)}+O(\varrho^2).$$

Now, (2.1) yields

$$\log\frac{f^*(z)}{f(z)} = \varrho e^{i\varphi}\left[\frac{f(z)}{f(z)-f(u)} + \frac{zf'(z)}{f(z)}\left(\frac{f(u)}{uf'(u)}\right)^2\frac{u}{u-z}\right] + \tag{2.3}$$
$$+\varrho e^{-i\varphi}\left[\frac{zf'(z)}{f(z)}\overline{\left(\frac{f(u)}{uf'(u)}\right)^2}\frac{\bar{u}z}{1-\bar{u}z}\right] + O(\varrho^2),$$

Using (2.2) and (2.3) we obtain

$$\delta\log\frac{f(z_1)}{f(z_2)} = \varrho e^{i\varphi}\left[\frac{f(u)}{uf'(u)}\right]^2\left[\frac{z_1f'(z_1)}{f(z_1)}\frac{u}{u-z_1} - \frac{z_2f'(z_2)}{f(z_2)} + \right. \tag{2.4}$$
$$\left. + \frac{f(z_1)}{f(z_1)-f(u)} - \frac{f(z_2)}{f(z_2)-f(u)}\right] + \varrho e^{-i\varphi}\overline{\left[\frac{f(u)}{uf'(u)}\right]^2}\left[\frac{z_1f'(z_1)}{f(z_1)}\frac{\bar{u}z_1}{1-\bar{u}z_1} - \right.$$
$$\left. - \frac{z_2f'(z_2)}{f(z_2)}\frac{\bar{u}z_2}{1-\bar{u}z_2}\right] = \varrho e^{i\varphi}S_1(u, z_1, z_2) + \varrho e^{-i\varphi}S_2(u, z_1, z_2).$$

The boundary points of $D(z_1, z_2)$ correspond to those functions $f(z)$ which yield stationary values of $\log|f(z_1)/f(z_2)|$ for fixed $\arg f(z_1)/f(z_2)$. Using the Lagrange multipliers we see that for the case of a local maximum of $|f(z_1)/f(z_2)|$ under the restriction $\arg f(z_1)/f(z_2) = \beta = \text{const.}$, there exists a real number $\lambda = \lambda(\beta)$ such that

$$\delta\left(\log\left|\frac{f(z_1)}{f(z_2)}\right| + \lambda\arg\frac{f(z_1)}{f(z_2)}\right) = \delta\left\{\Re\log\frac{f(z_1)}{f(z_2)} + \lambda\Im\log\frac{f(z_1)}{f(z_2)}\right\} \leqslant 0,$$

i.e.

$$\Re\left\{(1-i\lambda)\,\delta\log\frac{f(z_1)}{f(z_2)}\right\} \leqslant 0. \tag{2.5}$$

Using the equality $\Re(a+\bar{b}) = \Re(a+b)$, (2.4) and (2.5) we have

$$\Re\{(1-i\lambda)[\varrho e^{i\varphi}S_1(u, u_1, z_2) + \varrho e^{-i\varphi}S_2(u, z_1, z_2)]\} =$$
$$= \Re\{\varrho e^{i\varphi}[(1-i\lambda)S_1(u, z_1, z_2) + (1+i\lambda)\overline{S}_2(u, z_1, z_2)]\} \leqslant 0$$

for any $\varrho > 0$ and any real φ. This implies

$$(1-i\lambda)S_1(u, z_1, z_2) + (1+i\lambda)\overline{S}_2(u, z_1, z_2) = 0$$

and putting z instead of u, we obtain

$$(1-i\lambda)\left\{\left[\frac{f(z)}{zf'(z)}\right]^2\left[\frac{z_1f'(z_1)}{f(z_1)}\frac{z}{z-z_1} - \frac{z_2f'(z_2)}{f(z_2)}\frac{z}{z-z_2}\right] + \right. \tag{2.6}$$
$$\left. + \frac{f(z_1)}{f(z_1)-f(z)} - \frac{f(z_2)}{f(z_2)-f(z)}\right\} + (1+i\lambda)\left\{\left[\frac{f(z)}{zf'(z)}\right]^2\left[\overline{\left(\frac{z_1f'(z_1)}{f(z_1)}\right)}\frac{z\bar{z}_1}{1-z\bar{z}_1} - \right.\right.$$
$$\left.\left. - \overline{\left(\frac{z_2f'(z_2)}{f(z_2)}\right)}\frac{z\bar{z}_2}{1-z\bar{z}_2}\right]\right\} = 0.$$

The same equation holds for local minima of $|f(z_1)/f(z_2)|$ under fixed $\arg\{f(z_1)/f(z_2)\}$. Put now

$$A = (1-i\lambda)\frac{z_1 f'(z_1)}{f(z_1)}, \quad B = (1-i\lambda)\frac{z_2 f'(z_2)}{f(z_2)}, \tag{2.7}$$

$$w = f(z),\ w_k = f(z_k),\ k = 1, 2. \tag{2.8}$$

The equation (2.6) takes now the following form

$$\left(\frac{w}{z}\frac{dz}{dw}\right)^2\left(\frac{Az}{z-z_1}-\frac{Bz}{z-z_2}+\frac{\bar{A}z\bar{z}_1}{1-z\bar{z}_1}-\frac{\bar{B}z\bar{z}_2}{1-z\bar{z}_2}\right)+\frac{(1-i\lambda)w(w_2-w_1)}{(w_1-w)(w_2-w)} = 0,$$

resp.

$$\frac{P(z)}{z} = \frac{(1-i\lambda)(w_1-w_2)}{w(w-w_1)(w-w_2)}\left(\frac{dw}{dz}\right)^2, \tag{2.9}$$

where

$$P(z) = \frac{A}{z-z_1}+\frac{\bar{A}\bar{z}_1}{1-z\bar{z}_1}-\frac{B}{z-z_2}-\frac{\bar{B}\bar{z}_2}{1-z\bar{z}_2}. \tag{2.10}$$

3. The form of $P(z)$

Considering for small, real θ the function $f^*(z) = f(ze^{i\theta})$, we obtain the formula

$$\log\frac{f^*(z)}{f(z)} = \frac{izf'(z)}{f(z)}\theta + O(\theta^2). \tag{3.1}$$

Using (2.2) and (3.1) we have

$$\delta\log\frac{f(z_1)}{f(z_2)} = i\theta\left[\frac{z_1 f'(z_1)}{f(z_1)}-\frac{z_2 f'(z_2)}{f(z_2)}\right].$$

Hence (2.5), in view of (2.7), takes the form

$$\Re\{i\theta(A-B)\} = \mathscr{I}\{\theta(B-A)\} \leqslant 0$$

for both positive and negative θ which implies

$$\mathscr{I}A = \mathscr{I}B, \tag{3.2}$$

or

$$A-\bar{A} = B-\bar{B}. \tag{3.3}$$

The same equation holds for local minima of $|f(z_1)/f(z_2)|$ under fixed $\arg\{f(z_1)/f(z_2)\}$.

We now prove that $z\,P(z)$ is real on $|z| = 1$. The identity $\mathscr{I}(a+b) = \mathscr{I}(a-\bar{b})$ implies

$$\mathscr{I}\left\{\frac{Ae^{i\theta}}{e^{i\theta}-z_1}+\frac{\bar{A}\bar{z}_1e^{i\theta}}{1-e^{i\theta}\bar{z}_1}\right\}=\mathscr{I}\left\{\frac{A}{1-z_1e^{-i\theta}}-\frac{Az_1e^{-i\theta}}{1-z_1e^{-i\theta}}\right\}=\mathscr{I}(A)$$

and similarly

$$\mathscr{I}\left\{\frac{Be^{i\theta}}{e^{i\theta}-z_2}+\frac{\bar{B}\bar{z}_2e^{i\theta}}{1-e^{i\theta}\bar{z}_2}\right\}=\mathscr{I}(B),$$

and therefore, in view of (2.9) and (3.2), we have

$$\mathscr{I}\{e^{i\theta}P(e^{i\theta})\}=0 \text{ for real } \theta.$$

Since

$$P(z)=\frac{[(A-\bar{A})-(B-\bar{B})]\bar{z}_1\bar{z}_2z^3+\ (lower\ \ powers\ \ of\ z)}{(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)},$$

(3.3) implies that $P(z)$ has at most two finite roots. Besides, the principle of reflection implies $zP(z)=\overline{\bar{z}^{-1}P(\bar{z}^{-1})}$, hence both roots of $P(z)$ are symmetric w.r.t. $|z|=1$. Now, the r.h.s. of (2.9) does not vanish for any $z\in K$, so that $P(z)\neq 0$ for $z\in K$ and this means that both roots of $P(z)$ necessarily lie on $|z|=1$.

We next prove that $zP(z)$ has a constant sign on $|z|=1$. Suppose that $|f(z_1)/f(z_2)|$ attains for a function $f\in S$ a local maximum under the restriction that $\arg f(z_1)/f(z_2)$ is fixed and let Γ be the boundary of $f(K)$. In view of (2.9) Γ is a union of a finite number of analytic arcs. If $w\in\Gamma$ and $\zeta=\varphi(w)\in\partial K$ correspond to each other, then the function f^* mapping K on a domain which arises by the displacement $\varrho p(w)$ of points w on Γ along the outward normal, satisfies according to G. Julia [3] the following equation

$$\log\frac{f^*(z)}{f(z)}=\frac{1}{2\pi}\int_\Gamma\frac{zf'(z)}{f(z)}\,\frac{\zeta+z}{\zeta-z}\,|\varphi'(w)|^2\varrho p(w)ds_w+O(\varrho^2), \tag{3.4}$$

where $p(w)$ is a real and continuous function of $w\in\Gamma$ which vanishes in the neighbourhood of points for which $\varphi(w)$ ceases to be analytic, and ϱ is a real parameter. In view of (2.2) and (3.4) we have

(3.5)

$$\delta\log\frac{f(z_1)}{f(z_2)}=\frac{1}{2\pi}\int_\Gamma\left[\frac{z_1f'(z_1)}{f(z_1)}\,\frac{\zeta+z_1}{\zeta-z_1}-\frac{z_2f'(z_2)}{f(z_2)}\,\frac{\zeta+z_2}{\zeta-z_2}\right]|\varphi'(w)|^2\varrho p(w)ds_w.$$

Now, (2.5) and (3.5) imply

$$(3.6)\qquad \Re\left\{(1-i\lambda)\,\delta\log\frac{f(z_1)}{f(z_2)}\right\} = \frac{1}{2\pi}\int_{\Gamma}\Re\left(A\,\frac{\zeta+z_1}{\zeta-z_1}-B\,\frac{\zeta+z_2}{\zeta-z_2}\right)\times$$
$$\times|\varphi'(w)|^2\varrho p(w)\,ds_w \leqslant 0$$

for the case of a local maximum. This means that

$$(3.7)\qquad \Re\left\{A\,\frac{e^{i\theta}+z_1}{e^{i\theta}-z_1}-B\,\frac{e^{i\theta}+z_2}{e^{i\theta}-z_2}\right\}\geqslant 0$$

for any real θ. Suppose, on the contrary, that the l.h.s. in (3.7) is negative on an analytic arc $\gamma_0\subset\Gamma$. Taking a continuous function $p(w)$ which is negative on the open arc γ_0 and vanishes outside it on Γ, we obtain a positive variation in (2.5). At the same time this is an admissible variation of f since negative values of $p(w)$ involve a shrinking of $f(K)$. On the other hand (3.6) and (3.7) imply that the complementary set $\mathscr{C}f(K)$ has no interior points since otherwise a function $p(w)$ providing a positive variation in (3.6) could be constructed. Hence $f(K)$ is a slit domain.

Now, we have for real θ

$$\Re\left\{\frac{Ae^{i\theta}}{e^{i\theta}-z_1}+\frac{\bar{A}\bar{z}_1e^{i\theta}}{1-\bar{z}_1e^{i\theta}}\right\} = \Re\left\{A\,\frac{e^{i\theta}+z_1}{e^{i\theta}-z_1}\right\}$$

and in view of (2.10) and (3.7) we see that $\Re\{e^{i\theta}P(e^{i\theta})\}\geqslant 0$ in the case of a local maximum. Similarly $\Re\{e^{i\theta}P(e^{i\theta})\}\leqslant 0$ for those $f(z)$ which correspond to local minima. Thus we have proved that $zP(z)$ is real and of constant sign on $|z|=1$. This implies that both roots of $zP(z)$ situated on $|z|=1$ coincide and $P(z)$ has the form

$$(3.8)\qquad P(z) = \frac{C^{-1}\bar{\eta}(z-\eta)^2}{(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)}$$

where C is real and $|\eta|=1$.

Using (2.9) and (3.8) we obtain the differential equation (3.9) of functions which correspond to the boundary points of $D(z_1,z_2)$:

$$(3.9)\qquad \frac{e^{-i\alpha}(z-e^{-i\alpha})^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)} = \frac{C(1-i\lambda)(w_1-w_2)}{w(w-w_1)(w-w_2)}\left(\frac{dw}{dz}\right)^2,$$

(α, C, λ are real constants, $e^{i\alpha}=\eta$).

4. Solution of the equation (3.9)

The equation (3.9) is formally identical with the equation (3.1), [4], and we can adopt the argument used in [4] in order to solve (3.9).

With any real a we can associate the rational function

$$Q(z, a) = \frac{e^{-ia}(z-e^{ia})^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1 z)(1-\bar{z}_2 z)} \tag{4.1}$$

as well as three complex numbers

$$A_k = A_k(a) = \int_{\lambda_k} e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta, \quad k = 0, 1, 2, \tag{4.2}$$

where $\varphi(z)$ is the branch of $[z(z-z_1)(z-z_2)(1-\bar{z}_1 z(1-\bar{z}_2 z)]^{-1/2}$ chosen so that $e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta > 0$ on $|\xi| = 1$ for $\arg\zeta$ increasing in the interval $(a, a+2\pi)$. Besides, λ_k denotes here a loop joining η to z_k ($k = 0, 1, 2$; $z_0 = 0$), i.e. a cycle consisting of a small circle $C(z_k, \varepsilon)$ centre at z_k described in the positive direction and of a rectilinear segment described twice and joining $C(z_k, \varepsilon)$ to η whose prolongation contains z_k. The radius ε is chosen so that the only critical point of the integrand inside $C(z_k, \varepsilon)$ is the centre. If the open segment (η, z_k) contains critical point of the integrand, we replace suitable parts of (η, z_k) by small semicircles so as to leave critical points on the left side, when passing from η to z_k. We put next

$$\Omega_k = \Omega_k(a) = A_k - A_0, \quad k = 1, 2, \tag{4.3}$$

and

$$\vartheta = \vartheta(a) = e^{i\beta} \tag{4.4}$$

where $\beta = \beta(a)$ is defined by the equation

$$\int_{a}^{\beta} \sin\frac{1}{2}(\theta - a)|e^{i\theta} - z_1|^{-1}|e^{i\theta} - z_2|^{-1}d\theta = \tag{4.5}$$

$$= \int_{\beta}^{a+2\pi} \sin\frac{1}{2}(\theta - a)|e^{i\theta} - z_1|^{-1}|e^{i\theta} - z_2|^{-1}d\theta.$$

We have proved in [4] that $\mathscr{I}\{\Omega_2(a)/\Omega_1(a)\} \neq 0$. Therefore we may define the function

$$\tau(a) = \pm\Omega_2(a)/\Omega_1(a), \quad 0 \leqslant a < 2\pi, \tag{4.6}$$

where the sign is chosen so that $\mathscr{I}\tau(a) > 0$.

Besides, we have also proved that the functions

$$(4.7)\qquad F(z, a) = \mathfrak{p}\left[\int_{\vartheta(a)}^{z} \sqrt{Q(\zeta, a)}\, d\xi \,|\, \Omega_1(a), \Omega_2(a)\right]$$

are single-valued and univalent in the unit disc K for any real a and any path of integration situated inside K.

Putting

$$(4.8)\qquad v(z) = \int_{\vartheta(a)}^{z} \sqrt{Q(\zeta, a)}\, d\varsigma,$$

$$(4.9)\qquad w = 4C(1-i\lambda)(w_1 - w_2)W + \frac{1}{3}(w_1 + w_2),$$

we see that the equation (3.9) may be brought to the form $(dW/dv)^2 = 4W^3 - g_2 W - g_3$, where g_2, g_3 are constant and $w = W = \infty$ for $v = 0$. Hence $W = W(z) = \mathfrak{p}[v(z) \,|\, \omega', \omega'']$. It follows from the discussion of sections 2 and 3 in [4] that $W(z)$ represents a univalent and single-valued slit mapping if and only, if the lattices $m_1\omega' + m_2\omega''$, $m_1\Omega_1 + m_2\Omega_2$ are identical. This means that $W(z) = F(z, a)$, where $F(z, a)$ is defined by (4.7). In view of (4.9) we see that

$$(4.10)\qquad w = C_1 F(z, a) + C_2 = C_1\{F(z, a) + e_1(a) + e_2(a)\},$$

where C_1, C_2 are constant and

$$(4.11)\qquad e_k(a) = \mathfrak{p}\left[\frac{1}{2}\Omega_k(a)\right], \quad k = 1, 2.$$

If $f(z)$ is the function corresponding to a boundary point of $D(z_1, z_2)$. then the same consideration as that used in sect. 3, [4], yields

$$(4.12)\qquad f(z_1)/f(z_2) = \lambda[\tau(a) + 1]$$

where $\lambda(\tau)$ is the Jacobian modular function and $\tau(a)$ is defined by (4.6),

5. The proof of the main result

Theorem. If G_k, H_k $(k = 1, 2)$ are defined by (1.1), then both circles defined by (1.2) have no points in common with the real axis $\mathscr{I}\xi = 0$. If $\gamma(z_1, z_2)$ is this circle which is situated in the upper half-plane $\mathscr{I}\xi > 0$, then all the boundary points of the region of variability of the ratio, $\{f(z_1)/f(z_2)\}$ have the form $\lambda(\xi(a))$, where $\zeta(a) \in \gamma(z_1, z_2)$ and $\lambda(\varsigma) = k^2(\varsigma)$ is the Jacobian modular function.

Proof. Suppose first that the points z_k, $k = 0, 1, 2$, are not collinear. Let $(0k)$, $k = 1, 2$, be the loop joining z_0 to z_k, i.e. a cycle consisting of two circles $C(z_0, \delta), C(z_k, \delta)$ of small radius δ and centres at z_0, z_k, both described in the positive direction and of a rectilinear segment described twice and joining both circles so that its prolongation contains z_0 and z_k. The radius δ is so small that the circles $C(z_k, \delta)$, $k = 0, 1, 2$, have no points in common and are all contained in the unit disc. In view of (4.12) it is sufficient to prove that

$$\lambda[1+\tau(a)] = \lambda[\zeta(a)] \tag{5.1}$$

with $\zeta(a)$ defined by (1.2) after a suitable choice of sign. The prolongations of the segments $[z_k, z_j]$, $j, k = 0, 1, 2$, divide the unit circle $|z| = 1$ into six arcs. For $\eta = e^{ia}$ situated on four of them both loops $(0k)$ are homotopic to the system of two loops λ_0, λ_k (defined in sect. 4) w.r.t. K punctured at z_j $(j \neq 0, k)$. We have therefore $\int\limits_{(0k)} \equiv \int\limits_{(0k)} e^{-ia/2}(\zeta - e^{ia}) \times \varphi(\xi)d\xi = A_k - A_0 = \Omega_k$ since after describing the loop λ_k the integrand changes the sign. Hence

$$1+\tau(a) = 1 \pm \Omega_2(a)/\Omega_1(a)$$

$$= 1 \mp \int\limits_{(02)} \Big/ \int\limits_{(01)} = 1 \mp \frac{e^{ia}\int\limits_{(02)}\varphi(\xi)d\xi - \int\limits_{(02)}\varsigma\varphi(\varsigma)d\varsigma}{e^{ia}\int\limits_{(01)}\varphi(\xi)d\xi - \int\limits_{(01)}\xi\varphi(\xi)d\xi} = 1 \mp \frac{e^{ia}G_2 - H_2}{e^{ia}G_1 - H_1} = \varsigma(a)$$

and (5.1) is proved in this case.

If $\eta = e^{ia}$ is situated on the arc of $|z| = 1$ whose end points are determined by the rays $[z_0, z_1]$, $[z_2, z_1]$, then the loop (01) is homotopic to the cycle $\lambda_0 + \lambda_1$, hence $\int\limits_{(01)} = A_1 - A_0 = \Omega_1$. On the other hand the loop (02) is homotopic to the cycle $\lambda_0 + \lambda_1 + \lambda_2 - \lambda_1$ w.r.t. K punctured at z_1. This implies $\varsigma(a) = 1 \mp \int\limits_{(02)} / \int\limits_{(01)} = 1 \pm (A_0 - 2A_1 + A_2)/(A_1 - A_0) = (1 \pm \Omega_2(a)/\Omega_1(a))2 \pm = [1+\tau(a)] \pm 2$. Since $\lambda(\tau)$ has the period 2, (5.1) holds also in this case.

Finally, on the sixth arc the loop (02) is homotopic to the cycle $\lambda_0 + \lambda_2$ w.r.t. K punctured at z_1, whereas the loop (01) is homotopic to the cycle $\lambda_0 + \lambda_2 + \lambda_1 - \lambda_2$ w.r.t. K punctured at z_2 so that

$$\int\limits_{(02)} = A_2 - A_0 = \Omega_2, \quad \int\limits_{(01)} = A_0 - 2A_2 + A_1 = \Omega_1 - 2\Omega_2.$$

We have

$$\xi(a) = 1 \pm \int\limits_{(02)} \Big/ \int\limits_{(01)} = 1 \pm \frac{\Omega_2/\Omega_1}{1 - 2\Omega_2/\Omega_1}. \tag{5.2}$$

If $\mathscr{I}\{\Omega_2/\Omega_1\} > 0$, then $\tau = \Omega_2/\Omega_1$, and (5.2) takes the form $\xi(a) = 1 + \tau/(1-2\tau) = (1-\tau)/(1-2\tau)$. Putting $1+\tau = v$, we have $\xi(a) = (v-2)/(2v-3) = (av+b)/(ev+d)$, where $a \equiv d \equiv 1 \pmod 2$, $b \equiv c \equiv 0$

(mod 2), $ad-bc = 1$, which amens that $(av+b)/(cv+d)$ is a modular transformation. The automorphic property of λ implies $\lambda\left(\frac{v-2}{2v-3}\right) = \lambda(v)$ and (5.1) follows.

If $\mathscr{I}\{\Omega_2/\Omega_1\} < 0$, then $\tau = -\Omega_2/\Omega_1$. We have in this case from (5.2): $\varsigma(a) = 1\pm\tau/(1+2\tau) = 1+\tau/(1+2\tau)$. Putting $1+\tau = v$, we obtain $1+$ $+\tau/(1+2\tau) = (3v-2)/2v-1)$ which is another modular transformation. Hence $\lambda(v) = \lambda\left(\frac{3v-2}{2v-1}\right)$ and (5.1) follows again. The one-sided continuity proves our theorem in the case of z_k and η situated on one straight line. Since the automorphic transformations preserve the real axis, we have always $\mathscr{I}\xi(a) \neq 0$ for otherwise we would also have $\mathscr{I}\tau(a) = 0$ which is impossible as shown in [4].

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Streszczenie

Niech S będzie klasą funkcji $f(z) = z+a_2z^2+\ldots$ regularnych i jednolistnych w kole jednostkowym i niech $0, z_1, z_2$ będą trzema różnymi punktami tego koła. W pracy tej dowodzę metodami wariacyjnymi, że wszystkie punkty brzegowe obszaru zmienności $D(z_1, z_2)$ stosunku $f(z_1)/$ $/f(z_2)$ przy f zmieniających się w klasie S leżą na krzywej analitycznej $\Gamma(z_1, z_2)$ będącej obrazem okręgu o równaniu (1.2), leżącego w górnej półpłaszczyźnie, poprzez funkcję modułową.

Резюме

Пусть S будет классом функций $f(z) = z+a_2z^2+\ldots$ регулярных и однолистных в единичном круге и пусть 0, z_1, z_2 будут три разные точки этого круга. В этой работе доказывается, что все граничные точки области $D(z_1, z_2)$ всех возможных значений отношения $f(z_1)/f(z_2)$, если z_1, z_2 фиксированы, а f изменяется в классе S, лежат на аналитической кривой $\Gamma(z_1, z_2)$, которая является образом круга (1.2) верхней полуплоскости при преобразовании модулярной функции Якоби.