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On a Problem of M. Biernacki

O pewnym problemie M. Biernackiego

Об одной проблеме М. Бернацкого

Introduction. Let S_1 be the class of functions which are regular and univalent in the unit disc and vanish at the origin. M. Biernacki put the problem of evaluating the exact value r_u of the radius of univalence of the integral $\int_0^z t^{-1}f(t)\,dt$ for $f \in S_1$. In [1] he proved that $r_u = 1$. However, the proof contains a flaw and, as pointed out in [3], $r_u < \tanh \pi < 1$. On the other hand it is well known that for a close — to — convex and even for a close — to — star function $f(z)$, the integral $\int_0^z t^{-1}f(t)\,dt$ is close — to — convex and hence univalent so that, in view of a result of J. Krzyż [4], $r_u > 0{,}8$. In this paper we show that $r_u > 0{,}91$.

1. Let $D_2(0)$ be the class of simply connected domains D of hyperbolic type which have the following property: for any $w \in \bar{D}$ (where $\bar{D}$ is the closure of D) there exists a polygonal line consisting of at most two segments joining w to 0 and contained in $\bar{D}$.

Let $r_2(0)$ be the greatest positive number such that for any $f \in S_1$ and any $\varrho < r_2(0)$ the image domain of $|z| < \varrho$ belongs to the class $D_2(0)$. G. M. Golusin [2] p. 00, showed that $r_2(0) > 0{,}91$.

In what follows we will need the following characteristic property of close-to-star functions. The function $f(z)$ is close-to-star in the unit disc if and only, if for any $r \in \langle 0, 1)$ and any real θ_1, θ_2 $(\theta_1 < \theta_2)$

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) > -\pi. \tag{1}$$

We now prove the following

Theorem 1. If $f \in S_1$ then $f(z)$ is close-to-star in the disc $|z| < r_2(0)$.

Proof. Suppose that $f \in S_1$ and that for a positive $r, r < r_2(0)$ the condition (1) does not hold. Hence we can find $\theta_1, \theta_2 (\theta_1 < \theta_2)$ for which

$$A(\theta_2) \leqslant A(\theta_1) - \pi, \tag{2}$$

with $A(\theta) = \arg f(re^{i\theta})$ varying in a continuous manner. In view of continuity we can find the smallest number φ_2 in $(\theta_1, \theta_2\rangle$ for which $A(\varphi_2) = A(\theta_1) - \pi$. Obviously $A(\theta) > A(\theta_1) - \pi$ for any $\theta \in \langle\theta_1, \varphi_2)$. We can also find $\varphi_1 \in (\theta_1, \varphi_2)$ such that $A(\varphi_1) = A(\theta_1)$, whereas $A(\theta) < A(\varphi_1)$ in $(\varphi_1, \varphi_2\rangle$. Hence for $\varphi_1 < \varphi_2$ and $\theta \in (\varphi_1, \varphi_2)$

$$A(\varphi_2) = A(\varphi_1) - \pi \tag{3}$$

$$A(\varphi_1) - \pi < A(\theta) < A(\varphi_1). \tag{4}$$

Let k be the straight line joining $w = 0$ to $w = f(re^{i\varphi_2})$. From (3) it follows that $w = 0$ lies inside the straight line segment $\langle A_1, A_2\rangle$, where $A_k = f(re^{i\varphi_k})$, $k = 1, 2$. On the other hand the arc C_0: $w = f(re^{i\theta})$, $\theta \in \langle\varphi_1, \varphi_2\rangle$ is situated in one halfplane with the edge k. Obviously no subarc of C_0 reduces to a straight line segment. Let Q be the point on C_0 with the greatest distance from k and let l be the tangent of C_0 at Q. Clearly l and k are parallel. The arc C_0 and the chord $\langle A_1, A_2\rangle$ bound a simply connected domain Ω such that the points of Ω sufficiently close to C_0 belong to the complementary set of D_r if D_r is the image domain of $|z| \leqslant r$ under $w = f(z)$. Each segment $\langle$Q, S$\rangle$, where S is a point of this half plane with the edge l which contains $w = 0$, has non-empty intersection with the complementary set of D_r in any neighbourhood of Q. If we join $w = 0$ to Q by a polygonal line, any point S on the segment emanating from Q either lies on l, or is situated in this halfplane with the edge l which does not contain $w = 0$. Now, the segment $\langle 0, \mathrm{S}\rangle$ intersects C_0 and has non-empty intersections with D_r and its complementary set which means that D_r does not belong to $D_2(0)$. This contradiction proves our theorem.

2. Theorem 2. The function $\varphi(z) = \int_0^z t^{-1} f(t)\,dt$ is univalent in the disc $|z| < r_2(0) = 0{,}91\ldots$ for any $f \in S_1$.

Proof. We have $z\varphi'(z) = f(z)$. In view of Theorem 1, $z\varphi'(z)$ is close-to-star in the disc $|z| < r_2(0)$. This means that $\varphi(z)$ is close-to-convex in the disc $|z| < r_2(0)$, [5], and our theorem is proved.

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Streszczenie

W pracy tej dowodzę, że dla każdej funkcji $f(z)$ holomorficznej i jednolistnej w kole $|z| < 1$ całka $\int_0^z t^{-1} f(t)\,dt$ jest funkcją jednolistną w kole $|z| < r_2(0) = 0{,}91\ldots$, gdzie $r_2(0)$ określone zostało na str. 39.

Резюме

В работе доказывается, что если $f(z)$ произвольная голоморфная и однолистная в круге $|z| < 1$ функция, тогда $\int_0^z t^{-1} f(t)\,dt$ является однолистной в круге $|z| < r_2(0) = 0{,}91\ldots$ где $r_2(0)$ точнее на стр. 39.