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Z Zakładu Funkcji Analitycznych na Wydziale Mat. Fiz. Chem. UMCS
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Some Remarks Concerning My Paper: On Univalent Functions with Two Preassigned Values

Pewne uwagi o mojej pracy: O funkcjach jednolistnych z dwiema zadanymi wartościami

Заметка о моей работе: Об однолистных функциях с двумя заданными значениями

1. Introduction, notations

A mistake committed in the formula (4.14) of [2], where the factor z_1-z_2 should be taken with the opposite sign, vitiates the argument of sect. 5 leading to the evaluation of a because the quadratic equation for $\eta = e^{ia}$ analogous to (5.3), [2], reduces to the identity $0 = 0$. Besides, the discussion concerning the single-valuedness of the extremal function, as well as its dependence on the homotopy classes of curves determining the periods Ω_k was incomplete, so that some supplementary remarks seem to be necessary. These drawbacks do not, however, affect those statements of [2] where the results of sect. 5 are not used and even the form (2.5), [2], of the univalent function maximizing the ratio $|F(z_1)/F(z_2)|$ remains true after replacing a by the right value $\bar{a}$ which can be found as follows.

With any real $a \in [0, 2\pi]$ we can associate the function

$$(1.1)\qquad Q(z, a) = \frac{e^{-ia}(z-e^{ia})^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1 z)(1-\bar{z}_2 z)},$$

as well as three complex numbers

$$(1.2)\qquad A_k = A_k(a) = \int_{\lambda_k} e^{-ia/2}(\zeta-e^{ia})\varphi(\zeta)d\zeta, \quad k = 0, 1, 2,$$

where $\varphi(z)$ is the branch of $[z(z-z_1)(z-z_2)(1-\bar{z}_1 z)(1-\bar{z}_2 z)]^{-1/2}$ chosen so that $e^{-ia/2}(\zeta-e^{ia})\varphi(\zeta)d\zeta > 0$ on $|\zeta| = 1$, for $\arg\zeta$ increasing in the interval $(a, a+2\pi)$. Here λ_k denotes a loop joining $\eta = e^{ia}$ to z_k, $k = 0, 1, 2$, which are three different points of the unit disc, $z_0 = 0$. We call a loop joining η to z_k a cycle λ_k consisting of a small circle $C(z_k, \varepsilon)$ centre at z_k described in the positive direction and of a rectilinear segment described twice and joining $C(z_k, \varepsilon)$ to η whose prolongation contains z_k. The radius ε is chosen so that the only critical point of the integrand inside $C(z_k, \varepsilon)$ is the centre. If the segment (η, z_k) contains critical points of the integrand, we replace suitable parts of (η, z_k) by small semicircles so as to leave critical points on the left side, when passing from η to z_k.

We put next

(1.3) $$\Omega_k = \Omega_k(a) = A_k - A_0, \quad k = 1, 2,$$

and

(1.4) $$\vartheta = \vartheta(a) = e^{i\beta},$$

where $\beta = \beta(a)$ is defined by the equation

(1.5) $$\int\limits_a^\beta \sin\frac{1}{2}(\theta - a)|e^{i\theta} - z_1|^{-1}\cdot|e^{i\theta} - z_2|^{-1}d\theta =$$

$$= \int\limits_\beta^{a+2\pi} \sin\frac{1}{2}(\theta - a)|e^{i\theta} - z_1|^{-1}\cdot|e^{i\theta} - z_2|^{-1}d\theta.$$

Hence $\eta = e^{ia}$ and $\vartheta = e^{i\beta}$ divide the circumference $|z| = 1$ into two arcs with common end points and of the same length $l(a)$ in the metric $|Q(z, a)|^{1/2}\cdot|dz|$. If

(1.6) $$\tau = \tau(a) = \mp\Omega_2(a)/\Omega_1(a),$$

where the sign is chosen so that $\mathfrak{I}(\tau) > 0$, then the right value $\tilde{a}$ maximizes the expression $|\lambda(\tau(a)+1)|$; $\lambda(\tau)$ denotes here the elliptic modular function (the Jacobian modulus) defined by equations:

$$\tau = iK(1-\lambda)/K(\lambda),$$

$$K(\lambda) = \int\limits_0^1 [(1-t^2)(1-\lambda t^2)]^{-1/2}dt,$$

where $K(\lambda)$ is real and positive for $0 < \lambda < 1$, cf. [3], p. 318.

If $\wp(v \mid \Omega_1, \Omega_2)$ is the Weierstrass's $\wp$ function with periods Ω_1, Ω_2 then the functions

(1.7) $$F(z, a) = \wp\left[\int\limits_{\vartheta(a)}^{z} \sqrt{Q(z, a)}d\zeta \mid \Omega_1(a), \Omega_2(a)\right]$$

are single-valued and univalent in the unit circle K for any real a and any path of integration situated inside K. The extremal function yielding the maximum $k(z_1, z_2)$ of the ratio $|F(z_1)/F(z_2)|$ within the family of functions $F(z)$ regular and univalent in K and vanishing at the origin, has the form

$$
(1.8)\qquad f(z) = C_1 F(z, \tilde{a}) + C_2 = \\
= C_1 \left\{ \wp \left[\int_0^z \sqrt{Q(\zeta, \tilde{a})}\, d\zeta + \frac{1}{2}\Omega_1(\tilde{a}) + \right.\right. \\
\left.\left. + \frac{1}{2}\Omega_2(\tilde{a}) \right] + e_1(\tilde{a}) + e_2(\tilde{a}) \right\}
$$

where

$$
(1.9)\qquad e_k(\tilde{a}) = \wp\left(\frac{1}{2}\Omega_k(\tilde{a})\right), \quad k = 1, 2,
$$

$\wp$ has periods $\Omega_1(\tilde{a})$, $\Omega_2(\tilde{a})$ and C_1, C_2 are constant. The periods $\Omega_1(\tilde{a})$, $\Omega_2(\tilde{a})$ may be replaced by another pair of primitive periods $\omega_1(\tilde{a})$, $\omega_2(\tilde{a})$, with $\omega_1(\tilde{a})$ real and positive.

Besides, the map of K under $f(z)$ is a slit domain with the slit arising by a homothety from the map of a segment $[0, l(\tilde{a})]$ of the real axis under $\wp(x \mid \omega_1, \omega_2)$, where ω_1 is real. Finally

$$
(1.10)\qquad k(z_1, z_2) = \left|\lambda(\tau(\tilde{a}) + 1)\right|.
$$

2. The properties of the integral $\int_{\vartheta(a)}^{z} \sqrt{Q(\zeta, a)}\, d\zeta$

We first prove that $\tau(a)$ as defined by (1.6) cannot be real for any $a \in [0, 2\pi)$.

Similarly as in [4], p. 321, we see that the values $I = I(\Gamma)$ of the Abelian integral $\int_\Gamma e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)\, d\zeta$ taken along a closed curve Γ starting at $\eta = e^{ia}$ and situated inside K, have the form

$$
(2.1)\qquad I = I(\Gamma) = \sum_{k=0}^{2} \mu_k A_k
$$

where μ_k are integers which can assume arbitrary values for Γ suitably chosen and A_k are defined by (1.2). There are two possible cases: either all A_k are collinear (in this case all the values I lie on a straight line through the origin), or there exists a „lattice" of parallelograms covering all the plane such that to each corner point w there corresponds a curve Γ with $w = I(\Gamma)$. On the other hand we have also (cf. [4], p. 323)

(2.2) $$I = \varepsilon A_0 + m_1 \Omega_1 + m_2 \Omega_2,$$

where $\varepsilon = 0,1$ and m_1, m_2 are integers. Hence, if Ω_1, Ω_2 are collinear, the values I necessarily lie on a straight line through the origin and this means that also A_0, Ω_1, Ω_2 are collinear. Now, the circumference $|z| = 1$ may be deformed continuously into a system of three loops λ_r. After running around z_r on λ_r we come back to η with the opposite sign of $\varphi(z)$, hence $2l(a) = A_j - A_k + A_l = (A_j - A_0) - (A_k - A_0) + (A_l - A_0) + A_0$. Here and in what follows j, k, l are supposed to be three integers different from each other and taking the values 0, 1, 2; $z_0 = 0$. Therefore

(2.3) $$2l(a) = A_0 \mp \Omega_1 \mp \Omega_2 > 0,$$

where the signs depend on the relative position of η and z_k. This implies that all the numbers A_0, Ω_1, Ω_2, when collinear, must be real. In absence of poles of order higher than 1, after removing from the unit disc K the trajectories of the quadratic differential $Q(z, a)dz^2$ emanating from poles and zeros, we obtain a ring domain. Thus there exists a trajectory Γ_k of $Q(z, a)dz^2$ joining η to z_k and also a trajectory Γ_{jl} joining z_j to z_l. The orthogonal trajectory $\tilde{\Gamma}_l$ of $Q(z, a)dz^2$ starting at z_l attains $\partial K \cup \Gamma_k$ and for a cycle which can be shrinked continuously into $\tilde{\Gamma}_l$ plus a suitable arc of $\partial K \cup \Gamma_k$ emanating from η, we have $\frac{1}{2}|\Im I(\Gamma)| > 0$ since this gives the length of $\tilde{\Gamma}_l$ in the metric $|Q(z, a)|^{1/2}|dz|$. Hence $\Omega_1(a)$, $\Omega_2(a)$ cannot be both real and this proves that $\Im\tau(a) \neq 0$ for any real a.

Let now $I_0(z)$, $z \epsilon K$, be the value of $\int e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta$ taken along the segment $[\eta, z]$ with the points z_k possibly omitted along small semicircles. For any path joining η to z and situated in K we have either

(2.4) $$\int_\eta^z e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta = I_0(z) + m_1\Omega_1 + m_2\Omega_2,$$

or

(2.5) $$\int_\eta^z e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta = A_0 - I_0(z) + m_1\Omega_1 + m_2\Omega_2$$

where m_1, m_2 are integers, cf. [4], p. 324. Now, $\int_\eta^z = l(a) + \int_\theta^z$ and using this, (2.3), (2.4) and (2.5) we see that

(2.6) $$\int_{\theta(a)}^z e^{-ia/2}(\zeta - e^{ia})\varphi(\zeta)d\zeta = \mp[l(a) - I_0(z)] + m_1\Omega_1 + m_2\Omega_2,$$

where $\Im(\Omega_2/\Omega_1) \neq 0$.

This implies that the functions $F(z, a)$ as defined by (1.7) are single-valued and regular in K for any real a.

We now prove that the periods $\Omega_1(a)$, $\Omega_2(a)$ may be replaced by another pair $\omega_1(a)$, $\omega_2(a)$ of primitive periods with $\omega_1(a)$ real.

There are two possible cases.

(i) A trajectory γ separating $\partial K \cup \Gamma_k$ from the trajectory Γ_{jl} joining z_k to z_l can be deformed in a continuous manner into a system of two loops λ_j, λ_l joining η to z_j and z_l resp. Since $\varphi(z)$ changes the sign after running around z_l, we have

$$(2.7)\qquad \int_\gamma \sqrt{Q(\zeta, a)}\,d\zeta = \mp(A_j - A_l) = \mp(A_j - A_0) - (A_l - A_0) = \text{a real number}$$

and this means that one of the numbers $\Omega_1 - \Omega_2$ $(k = 0)$, Ω_1 $(j = 1, l = 0)$, Ω_2 $(j = 0, l = 1)$ is real. In the first case we may put $\omega_1 = \Omega_1 - \Omega_2$, $\omega_2 = \Omega_2$ and so we obtain the same lattice of periods with one real period.

(ii) If the trajectory γ cannot be continuously deformed into a system of loops λ_k, λ_l, we have

$$(2.8)\qquad \int_\gamma \sqrt{Q(\zeta, a)}\,d\zeta = A_j - 2A_k + A_l = (A_j - A_0) + (A_l - A_0) - 2(A_k - A_0) = \text{a real number}$$

This means that one of the following numbers is real: $\Omega_1 + \Omega_2$ $(k = 0)$, $2\Omega_1 - \Omega_2$ $(k = 1)$, $2\Omega_2 - \Omega_1$ $(k = 2)$. Putting $\omega_1 = \Omega_1 + \Omega_2$, $\omega_2 = -\Omega_1$; $\omega_1 = 2\Omega_1 - \Omega_2$, $\omega_2 = \Omega_1$; $\omega_1 = 2\Omega_2 - \Omega_1$, $\omega_2 = -\Omega_2$ we obtain in each case the same lattice of periods with ω_1 real. We may suppose that the real primitive period ω_1 is positive and then it represents according to (2.7) and (2.8) the length of γ in the metric $|Q(z, a)|^{1/2}|dz|$.

In all cases considered there exists another primitive period ω_2 of the form $\omega_2 = \Omega_j$, $j = 1, 2$, and $\frac{1}{2}|\Im\Omega_j|$ is the length of arcs of orthogonal trajectories joining Γ_{jl} to $\partial K \cup \Gamma_k$.

We now prove that the functions $F(z, a)$ defined by (1.7) are single valued and univalent in the unit circle.

We find on trajectories separating Γ_{jl} from $\partial K \cup \Gamma_k$ points whose distance from the orthogonal trajectory starting at ϑ and attaining Γ_{jl} measured in the metric $|Q(z, a)|^{1/2}|dz|$ along trajectories is equal $\frac{1}{2}\omega_1$. We obtain in this way an orthogonal trajectory $\tilde{\Gamma}_k$ emanating from z_k.

Now, open arcs of trajectories sweep out the domain $K-(\Gamma_k\cup\Gamma_{jl}\cup\tilde{\Gamma}_k)$ and each arc is mapped under $v(z)=\int_{\vartheta}^{z}\sqrt{Q(\zeta,a)}\,d\zeta$ on an open straight line segment $|\Re v|<\frac{1}{2}\omega_1$, $\Im v=\text{const}$ in a biunivoque manner.

Thus the mapping $v(z)=\int_{\vartheta}^{z}\sqrt{Q(\zeta,a)}\,d\zeta$ carries 1:1 the unit disc with removed closed arcs Γ_k, Γ_{jl}, $\tilde{\Gamma}_k$ into the rectangle

$$-\frac{1}{2}\omega_1<\Re v<\frac{1}{2}\omega_1,\quad 0<\Im v<\frac{1}{2}\omega_2, \tag{2.9}$$

where ω_1 is real and positive and the lattices $m_1\omega_1+m_2\omega_2$, $m_1\Omega_1+m_2\Omega_2$ are identical. Since $\wp(v\mid\omega_1,\omega_2)$ is an even elliptic function of order 2, it is univalent in the rectangle (2.9). This and the formula (2.6) imply that $F(z,a)$ are functions regular and univalent in the unit circle for any real a. Besides, the map of K under $F(z,a)$ is a slit domain because $\wp$ takes in the closure of the rectangle (2.9) every value. For $z\in\partial K$, $v(z)$ is real, hence the slit is the image of $[0,l(a)]$ under $F(z,a)$.

3. Determination of $\tilde{a}$

As shown in [2], the univalent function $w=f(z)$ for which $|f(z_1)/f(z_2)|=\sup|F(z_1)/F(z_2)|$,the least upper bound being taken with respect to functions $F(z)$ regular and univalent in the unit circle and vanishing at the origin, satisfies the differential equation

$$Q(z,a)=\frac{C(w_2-w_1)}{w(w_1-w)(w_2-w)}\left(\frac{dw}{dz}\right)^2, \tag{3.1}$$

where C is a real and positive constant, a is real, $w_k=f(z_k)$, $k=1,2$, and $Q(z,a)$ is defined by (1.1). Besides, $f(z)$ maps K on the w-plane slit along an analytic arc joining $f(\eta)$ to $f(\vartheta)=\infty$. Putting

$$v(z)=\int_{\vartheta(a)}^{z}\sqrt{Q(\zeta,a)}\,d\zeta, \tag{3.2}$$

$$w=4C(w_2-w_1)W+\frac{1}{3}(w_1+w_2), \tag{3.3}$$

we see that (3.1) is equivalent to $(dW/dv)^2=4W^2-g_2W-g_3$, where g_2,g_3 are constant. Since $W=\infty$ for $v=0$, resp. $z=\vartheta(a)$, we have necessarily $W(z)=\wp\big(v(z)\mid\omega',\omega''\big)$. It follows from the discussion of sect. 2 that $W(z)$ represents a univalent and single-valued slit mapping if and only, if the lattices $m_1\omega'+m_2\omega''$, $m_1\Omega_1+m_2\Omega_2$ are identical. This means that $W(z)=F(z,a)$, and hence $f(z)=C_1F(z,a)+C_2$ where C_1, C_2 are constant. Putting $z=z_k$ in (2.6) and using the equality $A_k=2I_0(z_k)$, we obtain for $k=0,1,2$

$$\int_{\theta(a)}^{z_k}\sqrt{Q(\zeta,a)}\,d\zeta=\mp\left[l(a)-\frac{1}{2}A_k\right]+m_1\Omega_1+m_2\Omega_2. \tag{3.4}$$

In view of (2.3) we see that

$$\int_{\theta(a)}^{0}\sqrt{Q(\zeta,a)}\,d\zeta=\frac{1}{2}(\mp\Omega_1\mp\Omega_2)+m_1\Omega_1+m_2\Omega_2, \tag{3.5}$$

or $0=f(0)=C_1\wp\left[\frac{1}{2}(\Omega_1+\Omega_2)\right]+C_2=C_1e_3(a)+C_2$, where

$$e_3(a)=\wp\left[\frac{1}{2}\big(\Omega_1(a)+\Omega_2(a)\big)\right]=-e_1(a)-e_2(a). \tag{3.6}$$

Thus $f(z)$ has the form

$$f(z)=C_1\{F(z,a)-e_3(a)\}=C_1\{F(z,a)+e_1(a)+e_2(a)\} \tag{3.7}$$

where a is a real and C_1 a complex constant. From (3.7), (1.7) and (3.4) we have for $k\neq l$, $k,l=1,2$: $f(z_k)=C_1\left\{\wp\left[l(a)-\frac{1}{2}A_k\right]-e_3(a)\right\}$ and using (2.3) we obtain $f(z_k)=C_1\left\{\wp\left[\left(l(a)-\frac{1}{2}A_0\right)-\frac{1}{2}(A_k-A_0)\right]-e_3\right\}=$ $=C_1\left\{\wp\left[\frac{1}{2}(\mp\Omega_1\mp\Omega_2)-\frac{1}{2}\Omega_k\right]-e_3(a)\right\}=C_1\left\{\wp\left[\frac{1}{2}\Omega_l\right]-e_3(a)\right\}=C_1[e_l(a)-$ $-e_3(a)]$. Hence

$$f(z_1)/f(z_2)=[e_2(a)-e_3(a)]/[e_1(a)-e_3(a)]. \tag{3.8}$$

It is well known, cf. e. g. [1], p. 178, that the expression $(e_3-e_2)/(e_1-e_2)$, where e_k are defined by (1.9) (with a instead of $\tilde{a}$) and (3.6) is equal to the Jacobian modulus $\lambda(\tau)$, τ being defined by (1.6). Hence (3.8) takes the form $f(z_1)/f(z_2)=\lambda(\tau)/[\lambda(\tau)-1]$. Putting $\omega_1=\Omega_1$, $\omega_2=\Omega_1+\Omega_2$ we obtain another pair of primitive periods with $\omega_2/\omega_1=1+\tau$. If $E_k=\wp\left(\frac{1}{2}\omega_k\right)$, $k=1,2$, $E_3=\wp\left[\frac{1}{2}(\omega_1+\omega_2)\right]$, then $\lambda(1+\tau)=(E_3-$ $-E_2)/(E_1-E_2)=(e_2-e_3)/e_1-e_3)$.

In view of (3.8) we see that $|f(z_1)/f(z_2)|=|\lambda(\tau(a)+1)|$ and hence a must be chosen so as to maximize the latter expression. This implies

(1.10) and the form

$$f(z) = C_1[F(z, \tilde{a}) + e_1(\tilde{a}) + e_2(\tilde{a})] \tag{3.8}$$

of the extremal function. We can eliminate $\vartheta(\tilde{a})$ from (3.8) deforming the path of integration so that it passes through the origin. We have $\int_{\vartheta(a)}^{z} = \int_{\vartheta(a)}^{0} + \int_{0}^{z}$ and in view of (3.5) we obtain the second form of the extremal function as given in (1.8).

We have $\Re\{\tau'(a)\lambda'(\tau+1)/\lambda(\tau+1)\} = 0$ in the extremal case, and using this and the identity

$$\lambda(\tau+1) = -16q\prod_{n=1}^{\infty}(1+q^{2n})^8(1-q^{2n-1})^{-8},$$

where $q = e^{\pi i\tau}$, cf. [3], p. 319, we can easily obtain a transcendental equation for $\tilde{a}$.

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Streszczenie

W pracy tej dokonano modyfikacji niektórych wyników pracy poprzedniej [2], będących konsekwencjami równania (5.3). Równanie to zostało wyprowadzone ze wzoru (4.14) w pracy [2] ,w którym czynnik $z_1 - z_2$ został omyłkowo wzięty ze znakiem przeciwnym.

Резюме

В этой работе дается модификация некоторых результатов работы [2] будущих следствиями уравнения (5. 3), которое получилось из формулы (4. 14) в работе [2], где умножитель $z_1 - z_2$ оказался ошибочно взятый с обратным знаком.