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Starlike Majorants and Subordination

Majoranty gwiaździste a podporządkowanie

Звездообразные мажоранты и подчинение

1. Introduction. This paper is a continuation of an earlier note [3] dealing with the connexion between majorization and subordination and we will use some definitions and notations introduced in [3]. The aim of this paper is to give a complete solution of the problem dealt with in [3] for the particular case of starlike majorants. We will prove the following

Theorem A. *Suppose* $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 \geqslant 0$ *and* $F(z) = z + A_2 z^2 + \ldots$, *are functions regular in the unit circle* C_1, $(|z| < 1)$ *such that* $|f(z)| \leqslant |F(z)|$ *for* $z \in C_1$, *and suppose moreover that* $F(z)$ *is starlike with respect to the origin* (i. e. $F \in S^*$). *Then for every* ϱ, $0 < \varrho \leqslant R^*$ *we have* $f(C_\varrho) \subset F(C_\varrho)$, *where* $C_\varrho = \{z : |z| < \varrho\}$, *and* R^* *is the unique real root of the equation* $x^3 + x^2 + 3x - 1 = 0$. *The number* R^* *is best possible and does not depend on* f, *or* F.

This theorem can be considered as a converse of the following result due to Golusin: suppose $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 \geqslant 0$ and $F(z)$ are regular in C_1, $F \in S^*$ and $f(C_1) \subset F(C_1)$. Then $|f(z)| \leqslant |F(z)|$ for $z \in C_{r^*}$, where the constant $r^* = \frac{1}{2}(3 - \sqrt{5}) = 0{,}386\ldots$ is best possible [1]. This paper contains the proof of the first part of Theorem A. The proof of the second part will be omitted here because it is contained in [3], where for any $R > R^*$ two functions f, F are constructed which satisfy all the hypotheses of Theorem A apart from $f(C_R) \subset F(C_R)$. Theorem A could be restated in a slightly more general form as follows.

Theorem B. *Suppose* $f(z) = a_1z+a_2z^2+\ldots$ *and* $F(z) = A_1z+A_2z^2+\ldots$ *are regular in* C_1, *F being univalent and starlike w. r. t. the origin in* C_1, $\arg a_1 = \arg A_1$, *unless* $a_1 = 0$, and $|f(z)| \leqslant |F(z)|$ *for* $z \in C_1$. *Then* $f(C_\varrho) \subset F(C_\varrho)$ *for any* $0 < \varrho \leqslant R^*$. R^* *does not depend on f or F and we can take as* R^* *the unique real root of the equation* $x^3+x^2+3x-1 = 0$. *Then* R^* *cannot be replaced by any greater number.*

The Theorem B can be obtained from the Theorem A by a suitable homothety and rotation.

2. The domain Ω_r and the classes $N(a)$, N.

Let C_r^0 be the boundary of C_r and Ω_r $(0 \leqslant r < 1)$ a domain whose boundary consists of the left-hand half of C_r^0 and two circular arcs symmetric to each other w. r. t. the real axis, intersecting each other at $z = 1$ and touching C_r^0 at $z = \pm ir$.

A function $\omega(z)$ regular in C_1 is said to belong to the class $N(a)$, $0 \leqslant a \leqslant 1$, if $\omega(0) = a$ and $|\omega(z)| \leqslant 1$ in C_1. Let N be the union of all classes $N(a)$, whereas a ranges over $\langle 0,1\rangle$. In view of Schwarz's lemma

$$\left|\frac{\omega(z)-a}{1-a\omega(z)}\right| \leqslant |z| \quad \text{for any} \quad \omega \in N(a). \tag{1}$$

This means that for fixed r and $|z| \leqslant r$ the values of $\omega(z)$, $\omega \in N(a)$, are inside the disc

$$\left|w - a\frac{1-r^2}{1-a^2r^2}\right| \leqslant r\frac{1-a^2}{1-a^2r^2}. \tag{2}$$

For any real φ the functions

$$\omega_\varphi(z) = (ze^{i\varphi}+a)/(1+aze^{i\varphi})$$

correspond to the points on the boundary of the circle (2). If $\omega_1, \omega_2 \in N(a)$, then $\lambda\omega_1+(1-\lambda)\omega_2 \in N(a)$, $(0 \leqslant \lambda \leqslant 1)$ and this implies that the values of $\omega(z)$ (z ranging over $\bar{C}_r$ and ω varying over $N(a)$) cover entirely the circle (2). If a ranges over $\langle 0,1\rangle$ the envelope of all the circles (2) is the boundary of Ω_r (see e. g. [2], p. 410). Hence we have

Lemma 1. *Suppose that z ranges over* $\bar{C}_r$ *(r fixed,* $0 < r < 1$*) and* ω *is changing within the class N. Then the points* $\omega(z)$ *cover the closure of* Ω_r.

3. The domain S_r^*.

According to [4], p. 123 for any fixed z_0, z_1 in C_1 the domain of possible values of $[z_1F(z_0)/z_0F(z_1)]^{\frac{1}{2}}$ for F ranging over S^* is the circle with the boundary $w = (1-z_1e^{it})/(1-z_0e^{it})$, $-\pi \leqslant t \leqslant \pi$. This implies that the values of $[F(z_0)/F(z_1)]^{\frac{1}{2}}$, $|z_0| = |z_1| = r < 1$ cover the disc $K(\varphi)$:

$$\zeta = e^{i\varphi/2}[(1-re^{it})/(1-re^{it}e^{i\varphi})], \quad -\pi \leqslant t \leqslant \pi,$$

where $\varphi = \arg(z_0/z_1)$. The equation of the circle $K(\varphi)$ has also the form

$$(3) \qquad \left|\frac{\zeta - e^{i\varphi/2}}{\zeta - e^{-i\varphi/2}}\right| = r.$$

We now determine the envelope of the circles (3) for varying φ. (3) has the form

$$\Phi(\zeta, \varphi) = \mathscr{R}\left\{\log\frac{\zeta - e^{i\varphi/2}}{\zeta - e^{-i\varphi/2}}\right\} - \log r = 0.$$

The equation $\partial\Phi/\partial\varphi = 0$ gives

$$\mathscr{R}\{e^{i\varphi/2}[(\zeta - e^{-i\varphi/2})/(\zeta - e^{i\varphi/2})]\} = 0$$

and in view of (3) we obtain the equation of the envelope:

$$(4) \qquad e^{i\varphi/2}\frac{\zeta - e^{-i\varphi/2}}{\zeta - e^{i\varphi/2}} = \pm\frac{i}{r}, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Taking the sign „+" we obtain the circumference L_r with the centre $S(r) = 2ri/(1-r^2)$ and the radius $\varrho(r) = (1+r^2)/(1-r^2)$ and taking the sign „$-$" we obtain the circumference L'_r symmetric to L_r with respect to the real axis. Both circumferences intersect at -1 and 1 (since $|1-S(r)| = \varrho(r)$) and enclose two closed discs K_r and K'_r respectively. Therefore all the possible values of $[F(z_0)/(F(z_1)]^{\frac{1}{2}}$, whenever $F \in S^*$, and z_0, z_1 satisfy $|z_0| = |z_1| = r$ cover the set being the closure of the symmetric difference: $\overline{K_r \dot{-} K'_r}$ of both circles K_r, K'_r. Hence the domain S^*_r covered by $F(z_0)/F(z_1)$ when z_0 and z_1 are moving on C^0_r and F is varying within S^* is identical with the set of all ζ^2, where ζ is ranging over the closure of $D_r = K_r - \mathrm{Int}(K'_r)$, resp. over the upper half of $\overline{K_r \dot{-} K'_r}$. Let a be the least distance of a point on L'_r from the origin and let $\eta = \eta(a, \varphi)$, $0 \leqslant \varphi \leqslant \pi$ be the equation of the upper part of L'_r in polar coordinates.

Corollary 1. *$\eta(a, \varphi)$, a being fixed, is a decreasing function of φ in $\langle 0, \pi/2\rangle$ and an increasing function of φ in $\langle\pi/2, \pi\rangle$, moreover $\eta(a, \pi/2) = a$*

It is easy to see that the distance of $S'(r)$ from the origin is equal $(1-a^2)/2a$ and hence from the triangle $OS'(r)A$ we have

$$(5) \qquad \eta(a, \varphi) = \frac{1}{2a}\{[(1-a^2)^2\sin^2\varphi + 4a^2]^{\frac{1}{2}} - (1-a^2)\sin\varphi\}.$$

We now prove the inequality

$$(6)\qquad \eta(h,\varphi) \leqslant \eta^2\left(\sqrt{h}, \frac{\varphi}{2}\right) \quad \text{for any } 0 \leqslant \varphi \leqslant \frac{\pi}{2},\ 0 < h < 1,$$

with the sign of equality for $\varphi = 0$ only.

It follows from (5) that (6) is equivalent with

$$(7)\qquad 2\{[(1-h^2)^2\sin^2\varphi + 4h^2]^{\frac{1}{2}} - (1-h^2)\sin\varphi\} \leqslant$$
$$\leqslant \left\{\left[(1-h)^2\sin^2\frac{\varphi}{2} + 4h\right]^{\frac{1}{2}} - (1-h)\sin\frac{\varphi}{2}\right\}^2,$$

and this gives

$$(8)\qquad (1-h)^2\sin^2\frac{\varphi}{2} + 4h \leqslant 4(1+h)^2\cos^2\frac{\varphi}{2}.$$

The left-hand side attains its greatest value $l = \frac{1}{2}(1-h)^2 + 4h$ in $\langle 0, \pi/2\rangle$ for $\varphi = \pi/2$, whereas the least value of the right-hand side in $\langle 0, \pi/2\rangle$ $p = 2(1+h)^2$ satisfies the inequality $l < p$, so that (8) and also (6) are proved.

Besides, for $0 < h < h' < 1$ we have evidently

$$(9)\qquad \eta(h,\varphi) < \eta(h',\varphi) \quad \text{for any } \varphi \in (0,\pi)$$

and, moreover

$$(10)\qquad \eta(h,0) = \eta(h,\pi) = 1$$

Let now G_a $(0 < a \leqslant 1)$ be the domain containing the origin and enclosed by the curve Γ_a:

$$(11)\qquad w = \eta^2(a, \theta/2)e^{i\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$$

It follows from (9) that for $a < a'$, $G_a \subset G_{a'}$, and in view of (10) we have

$$(12)\qquad \overline{G}_a - \{1\} \subset G_{a'}$$

We now prove

Lemma 2. *The boundaries of Ω_{a^2} and G_a have no points in common apart from $-a^2$ and 1.*

Proof. In view of Corollary 1 the point $-a^2$ is the nearest point on Γ_a from the origin. Hence the closed circle $\overline{C}_{a^2}$ is contained in G_a apart from the only point $-a^2$ where it is touching Γ_a from inside. The arc L'_{a^2} being a part of the boundary of Ω_{a^2} has the equation $w = \eta(a^2,\varphi)e^{i\varphi}$,

$0 \leqslant \varphi \leqslant \pi/2$. It follows from (6) with $h = a^2$, and from (11) that this arc, the point $w = 1$ excluded, lies entirely inside of Γ_a. Since G_a is symmetric w. r. t. the real axis, the remaining part of the boundary of Ω_{a^2} also lies inside of Γ_a and this proves our lemma.

In view of (12) we obtain

Corollary 2. *$\overline{\Omega}_{a^2} - \{1\} \subset G_{a'}$ for $0 < a < a' \leqslant 1$.*

It is easy to see that $a = a(r) = (1-r)/(1+r) = \varrho(r) - |S(r)|$. The domain $G_{a(r)}$ is contained in the complementary set of S_r^* by definitions of S_r^* and G_a.

4. Proof of the first part of the Theorem A.

Let us suppose, contrary to hypotheses, that there exist $r_0 \epsilon (0, R^*\rangle$ and two functions $f(z)$, $F(z)$, $F \epsilon S^*$, satisfying the assumptions of the Theorem A and such that $f \neq F$, $f(C_{r_0}) \subset F(C_{r_0})$. Thus for a value z_0, $|z_0| = \varrho_0 < r_0$, the point $f(z_0)$ is situated outside of the domain $F(C_{r_0})$ and so the segment $(0, f(z_0)$ contains a point $F(z_1)$ with $z_1 = \varrho_0$, in view of $\overline{F(C_{\varrho_0})} \subset F(C_{r_0})$. Put $\omega(z) = f(z)/F(z)$. We have $\omega(0) = a$, $0 \leqslant a < 1$, and $|\omega(z)| \leqslant 1$ in C_1. Hence

$$\frac{f(z_0)}{F(z_1)} = \frac{\omega(z_0) F(z_0)}{F(z_1)} > 1, \text{ or}$$

(13) $$\omega(z_0) = (1+\varepsilon)\frac{F(z_1)}{F(z_0)}, \quad \text{with } \varepsilon > 0.$$

It follows from Lemma 1 (in view of $f(z) \not\equiv F(z)$ which means $\omega(0) < 1$) that

(14) $$\omega(z_0) \epsilon \overline{\Omega}_{\varrho_0} - \{1\}.$$

$F(z_1)/F(z_0)$ is a point of $S_{\varrho_0}^*$ and this means $F(z_1)/F(z_0) \notin G_{a(\varrho_0)}$, and also for any $\varepsilon > 0$

(15) $$(1+\varepsilon)\frac{F(z_1)}{F(z_0)} \notin \overline{G}_{a(\varrho_0)}.$$

Now, R^* satisfies $x = (1-x)^2/(1+x)^2$, $0 < R^* < 1$, and hence for $\varrho_0 < R^*$ we have $\varrho_0 < [(1-\varrho_0)/(1+\varrho_0)]^2 = a^2(\varrho_0)$, or $\varrho_0^{1/2} < a(\varrho_0)$ and in view of Corollary 2

(16) $$\overline{\Omega}_{\varrho_0} - \{1\} \subset G_{a(\varrho_0)}.$$

(16), (14) and (13) imply

$$(17) \qquad (1+\varepsilon)\frac{F(z_1)}{F(z_0)}\in G_{a(\varrho_0)}$$

contrary to (15). Thus the first part of Theorem A is proved. Since the proof of the second part is contained in [3] the Theorem A is proved completely.

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Streszczenie

Dowodzi się, że jeśli $f(z) = a_1z+a_2z^2+\ldots$ i $F(z) = A_1z+A_2z^2+\ldots$ są funkcjami holomorficznymi dla $|z|<1$, a druga z nich jest jednolistna i gwiaździsta, jeśli $\arg a_1 = \arg A_1$ lub $a_1 = 0$ i jeśli $|f(z)| \leqslant |F(z)|$ dla $|z|<1$, wówczas obszar, na który funkcja $f(z)$ odwzorowuje koło $|z|<\varrho$, zawiera się zawsze w odpowiednim obszarze wartości funkcji $F(z)$, o ile tylko $0<\varrho\leqslant R^*$, gdzie R^* jest jedynym rzeczywistym pierwiastkiem równania $x^3+x^2+3x-1 = 0$ i nie może być zastąpione przez żadną liczbę większą.

Twierdzenie to jest poniekąd odwrotne do pewnego twierdzenia G. M. Goluzina [1]; tw. 8′.

Резюме

Доказывается, что если функции $f(z) = a_1z+a_2z^2+\ldots$ и $F(z) = A_1z+A_2z^2+\ldots$ голоморфны для $|z|<1$ причем вторая из них однолистная и звездная, если $\arg a_1 = \arg A_1$ или $a_1 = 0$ и если $|f(z)|\leqslant |F(z)|$ для $|z|<1$ тогда множество значений функций $f(z)$ для $|z|<\varrho$ заключается в соответствующим множестве значений второй функции $F(z)$ если только $0<\varrho\leqslant R^*$ где число R^* является единственным вещественным корнем уравнения $x^3+x^2+3x-1 = 0$ и уже не может быть увеличено.

В известной мере теорема эта является обратной к некоторой теореме Г. М. Голузина [1]; теор. 8′.