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On a Method to Deal with Convex Functions of Bounded Type

Pewna metoda badania funkcji o ograniczonej wypukłości

Convex functions of bounded type were introduced by Goodman in [1] and [2] in the following way:

For f regular and locally univalent in the unit disc $D = \{z|\ |z| < 1\}$

$$\kappa_f(z) := \frac{1}{|z f'(z)|} \operatorname{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right), \qquad 0 < |z| = r < 1 ,$$

is the curvature of $f(\{z|\ |z| = r\})$ at the point $f(z)$. f is said to be a member of the class $CV(R_1, R_2)$ if f is regular and convex in D, normalized by $f(0) = f'(0) - 1 = 0$ and

$$0 < R_1 \le \liminf_{|z|\to 1} 1/\kappa_f(z) \le \limsup_{|z|\to 1} 1/\kappa_f(z) \le R_2 < \infty .$$

In [1] and [2] a lot of problems concerning $CV(R_1, R_2)$ are discussed among them the coefficient problem and the question of the existence of a "nice" variational formula for $CV(R_1, R_2)$. In this paper we shall derive a simple method which enables us to get old and new inequalities for $CV(R_1, R_2)$ by similar operations. In fact, in these considerations we only use part of the conditions valid for $CV(R_1, R_2)$ and so we get results for the bigger class $C(1/R_2)$ defined below (compare [6]).

Definition 1. Let $K > 0$ and f be regular and locally univalent in D. Then f is said to belong to the class $C(K)$ if and only if

$$\liminf_{|z|\to 1} \kappa_f(z) \ge K .$$

Remark 1. As it was proved by Peschl in [3] and again discussed in [6] κ_f cannot have a local minimum in $D \setminus \{0\}$ if $f \in C(K)$ and so we see that the following assertion is valid: $f \in C(K)$, $K > 0$, if and only if $\kappa_f(z) > K$ for $z \in D \setminus \{0\}$. Another consequence of this fact needed in the sequel is as follows: If $f \in C(K)$, $K > 0$, and $f_r(z) = f(rz)$, $0 < r < 1$, then $f_r \in C(K') \subset C(K)$ for a $K' > K$ (compare [6]).

We prove:

Theorem 1. *Let $K > 0$, $f \in C(K)$ be regular and locally univalent in a disc $D_\varepsilon = \{z | \, |z| < 1 + \varepsilon\}$ for an $\varepsilon > 0$ and $t : \{z | \, |z| = 1\} \to \mathbb{R}$ continuous, nonnegative and not vanishing identically. Then for any $a \in D$ the inequality*

$$(1)\quad \begin{aligned} &\left|\frac{1}{2\pi}\int_0^{2\pi} K f'(e^{i\theta})(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})\, d\theta\right| \le \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\Big(1+\frac{e^{i\theta}f''(e^{i\theta})}{f'(e^{i\theta})}\Big)|1-\bar{a}e^{i\theta}|^2\, t(e^{i\theta})\, d\theta \end{aligned}$$

is valid and equality occurs if and only if

$$(2)\qquad f(z) = \frac{e^{i\phi}}{K}\,\frac{z-a}{1-\bar{a}z} + c\,, \qquad c \in \mathbf{C}\,,\ \phi \in \mathbb{R}\,.$$

Proof.

$$(3)\quad \left|\frac{1}{2\pi}\int_0^{2\pi} K f'(e^{i\theta})(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})\, d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} K|f'(e^{i\theta})|\,|1-\bar{a}e^{i\theta}|^2 t(e^{i\theta})\, d\theta$$

and equality occurs if and only if $\arg\big(f'(e^{i\theta})(1-\bar{a}e^{i\theta})^2\big)$ is constant for $\theta \in [0, 2\pi]$. Under the conditions of our theorem this implies according to the Schwarz reflection principle that $f'(z)(1-\bar{a}z)^2$ is the restriction to D_ε of a bounded entire function and therefore a constant.

$$\kappa_f(e^{i\theta}) = \frac{1}{|f'(e^{i\theta})|}\,\mathrm{Re}\Big(1+e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\Big) \ge K \qquad \text{for } \theta \in [0, 2\pi]$$

and (3) imply that (1) is valid. If we insert $f'(z) = d/(1-\bar{a}z)^2$ into (1) we see that equality can occur if and only if $K|d| = 1 - |a|^2$. This proves the rest of the assertion.

Remark 2. The procedure to get inequalities concerning $C(K)$ from Theorem 1 is the following: If $f \in C(K)$ then $f_r \in C(K)$ for $0 < r < 1$ and fulfills the conditions of Theorem 1. Now we choose

$$(4)\qquad t(z) = \prod_{j=1}^{n} \frac{(1-\bar{a}_j z)(z-a_j)}{(1-\bar{b}_j z)(z-b_j)}\,, \qquad a_j, b_j \in D$$

and evaluate the left side of (1)

$$L_a(f_r) = \left|\frac{1}{2\pi i}\int_{|z|=1} K f_r'(z)(1-\bar{a}z)^2\, t(z)\frac{dz}{z}\right|$$

by the Residue Theorem. The right side denoted by $R_a(f_r)$ in the sequel can be computed in many cases by use of the orthogonality of the trigonometric functions. As the last step let $r \to 1$. Naturally any of the inequalities got in this way is sharp for the functions (2) for any choice of a_j, b_j. At the end of the considerations one may choose special values for the parameters or consider the whole variety. We give some examples for this procedure.

Corollary 1. *Let $K > 0$ and $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$. For $a \in D \setminus \{0\}$ and any $b \in D$ the inequality*

$$K\left|f'(a)(1-|a|^2)(1-\bar{b}a)\frac{a-b}{a} + c_1\frac{b}{a}\right| \le 1 + |b|^2 - 2\,\mathrm{Re}\, b\frac{c_2}{c_1} \tag{5}$$

is valid. Euqality is attained if f is of the form (2).

Proof. We take

$$t(z) = \frac{(1-\bar{b}z)(z-b)}{(1-\bar{a}z)(z-a)}$$

and evaluate

$$L_a(f_r) = K\left|r\,f'(a)(1-|a|^2)(1-\bar{b}a)\frac{a-b}{a} + r\,c_1\frac{b}{a}\right|$$

and

$$R_a(f_r) = 1 + |b|^2 - 2\,\mathrm{Re}\, rb\frac{c_1}{c_2}\,.$$

The limiting process $r \to 1$ delivers (5).

Remark 3. For fixed values of c_1, c_2 and b (5) describes a disc $D(b)$, wherein $f'(a)$ can be situated, so $f'(a)$ has to lie in the intersection of the discs $D(b)$, $b \in D$, for fixed c_1 and c_2.

Now we choose special values for b:

Corollary 2. *Let $K > 0$, $f \in C(K)$. Then for each $a \in D$*

$$K|f'(a)|(1-|a|^2) \le 1\,. \tag{6}$$

Equality occurs for functions of the form (2).

Corollary 2 is an immediate consequence of (5) with $b = 0$ and was proved by Goodman in [2] by means of an area theorem.

Corollary 3. *Let $K > 0$ and $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$. Then*

$$\left|\frac{c_2}{c_1}\right|^2 \le 1 - K|c_1|\,. \tag{7}$$

Equality occurs for any function of type (2).

Proof. Let $b = a$ in (5) and choose $a \in D \setminus \{0\}$ such that $a\frac{c_2}{c_1} \ge 0$. Then (5) implies

$$\left|\frac{c_2}{c_1}\right| \le \frac{1+|a|^2 - K|c_1|}{2|a|} := F(|a|)\,. \tag{8}$$

If $1-K|c_1|>0$, $\min\limits_{0<|a|<1} F(|a|) = F(\sqrt{1-K|c_1|}) = \sqrt{1-K|c_1|}$.

If $1-K|c_1|=0$, $F(|a|)=|a|/2$ so that (8) implies $|c_2/c_1|=0$. Inequality (7) was conjectured by Goodman for $f \in CV(R_1,R_2)$ (see [1]) but in fact proved earlier by Raupach ([5]) using a method due to Peschl (see [4]). In [6] a proof of (7) may be found which is based on a minimum principle for locally univalent functions. We want to prove at this place that this inequality may serve to characterize $C(K)$ as the "linear invariant" family which fulfills (7):

Theorem 2. *Let $K>0$. $f(z)=\sum_{k=0}^{\infty} c_k z^k \in C(K)$ if and only if*

i) $f\left(\frac{z-a}{1-\bar{a}z}\right) \in C(K)$ *for any $a \in D$*

ii) $|c_2/c_1|^2 \le 1-K|c_1|$.

Proof. One direction of the proof is an immediate consequence of Corollary 3 and the geometric meaning of Definition 1. The other direction is proved as follows: Let

$$f\left(\frac{z-a}{1-\bar{a}z}\right) = \sum_{k=0}^{\infty} C_k(a) z^k .$$

Then according to i) and ii)

$$\left|\frac{C_2(a)}{C_1(a)}\right|^2 \le 1-K|C_1(a)| .$$

This implies

$$\frac{1}{|f'(a)|}\operatorname{Re}\left(1+a\frac{f''(a)}{f'(a)}\right) - \frac{1}{4|f'(a)|}\left|\frac{f''(a)}{f'(a)}\right|^2 (1-|a|^2) \ge K .$$

Hence

$$\liminf_{|z|\to 1} \kappa_f(a) \ge K$$

(compare [5] and [6]).

As an example involving Taylor coefficients of higher order we prove:

Corollary 4. *Let $K>0$, $n \ge 2$, $f(z)=\sum_{k=0}^{\infty} c_k z^k \in C(K)$, $1+z\frac{f''(z)}{f'(z)} = \sum_{k=0}^{\infty} E_k z^k$ and $a=\sqrt{1-K|c_1|}$. Then*

$$\begin{aligned} K\left|c_1 - \tfrac{1}{2}(n-1)c_{n-1}a^2 + n c_n a - \tfrac{1}{2}(n+1)c_{n+1}\right| &\le \\ \le 1+a^2 - a\operatorname{Re}E_1 - \tfrac{1}{2}(1+a^2)\operatorname{Re}E_n + \tfrac{1}{2}a\operatorname{Re}(E_{n-1}+E_{n+1}) &. \end{aligned} \tag{9}$$

Equality occurs for all functions of type (2) with $a \in [0,1)$.

Proof. Take $t(z) = 1-\frac{1}{2}(z^n+z^{-n})$

Remark 4. Inequalities of type (9) seem to be typical for the application of Theorem 1. Since Goodman proved in [1] and [2] that the sharp bound for $|c_k|$ in general is not attained for functions of type (2), it is clear that one cannot get the full information about the coefficient problem in this way. Nevertheless it may be worthwhile to try to get a good approximation varying the functions (4). We leave this for further investigations, since our aim in this paper was only to give an outline of the method.

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STRESZCZENIE

Niech $K > 0$ i niech f będzie funkcją regularną, lokalnie jednolistną w kole jednostkowym D. Mówimy, że f należy do klasy $C(K)$ funkcji o ograniczonej wypukłości, jeśli $\liminf\limits_{|z| \to 1} \kappa_f(z) \geq K$, gdzie $\kappa_f(z)$ oznacza krzywiznę w punkcie $f(z)$ warstwicy $w = f(|z|e^{i\theta})$, $0 \leq \theta \leq 2\pi$.

Dla funkcji $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ klasy $C(K)$ autor otrzymuje pewną nierówność całkową (tw.1), która ma szereg interesujących konsekwencji. Wykazano m.in., że rodzinę $C(K)$ można scharakteryzować jako rodzinę liniowo niezmienniczą funkcji spełniających nierówność $|c_2/c_1|^2 \leq 1 - K|c_1|$.

SUMMARY

Suppose $K > 0$ and let f be regular and locally univalent in the unit disk D. The function f is said to belong to the class $C(K)$ of functions of bounded convexity, if $\liminf\limits_{|z| \to 1} \kappa_f(z) \geq K$,

where $\kappa_f(z)$ denotes the curvature at $f(z)$ of the level line $w = f(|z|e^{i\theta})$, $0 \leq \theta \leq 2\pi$.

For $f(z) = \sum_{k=0}^{\infty} c_k z^k$ in $C(K)$ an integral inequality (Thm 1) has been derived and some interesting consequences are given. E.g. the class $C(K)$ can be characterized as the linearly invariant family of functions satisfying $|c_2/c_1|^2 \leq 1 - K|c_1|$.