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Geometrical Interpretation of the Frame Field along a Curve in the Space $P(p,q)$

Interpretacja geometryczna pola reperów wzdłuż krzywej w przestrzeni $P(p,q)$

In [5] we have constructed a linear orthonormal frame field along a curve in the space $P(p,q)$ of p-dimensional planes in the $(p+q)$-dimensional euclidean space E^{p+q}. The aim of the present paper is to prove some properties of vector fields of the constructed linear frame field and to give a geometrical interpretation of the striction line of a curve in $P(p,q)$.

Let us recall the theorem about the frame field along a curve in $P(p,q)$ and the definition of the striction line that are contained in [5].

Theorem 1. *Given an admissible oriented parametrized curve (a.o.p.c.) Σ in $P(p,q)$*

$$\Sigma : t \to [Y(t) : (\lambda^i) \to x(t) + \lambda^i e_i(t)] , \qquad i = 1,\dots,p$$

that is assumed to be neither r-cylindric for any $r = 1,\dots,p-1$ nor orthogonally s-cylindric for any $s = 1,\dots,q-1$.

(T,1) *There exists a linear orthonormal frame field in E^{p+q}*

$$\bar{A}(s) = [\bar{a}_1(s),\dots,\bar{a}_q(s),\bar{e}_1(s),\dots,\bar{e}_p(s),x(s)]$$

such that a parametrized curve (p.c.)

$$\bar{\Sigma} : s \to [\bar{Y}(s) : (\lambda^i) \to \bar{x}(s) + \lambda^i \bar{e}_i(s)]$$

and Σ determine the same curve $[\bar{\Sigma}] = [\Sigma]$ in $P(p,q)$.

(T,2)

$\bar{e}_1 := e_\circ$ – *the orienting directional vector of* Σ

$\bar{a}_1 := a_\circ$ – *the normal orienting directional vector of* Σ

$\bar{e}_2 := [(\dot{\bar{e}}_1)^2 - (\dot{\bar{e}}_1, \bar{a}_1)^2]^{-1/2} (\dot{\bar{e}}_1 - (\dot{\bar{e}}_1, \bar{a}_1)\bar{a}_1)$

$\bar{a}_2 := [(\dot{\bar{a}}_1)^2 - (\dot{\bar{a}}_1, \bar{e}_1)^2]^{-1/2} (\dot{\bar{a}}_1 - (\dot{\bar{a}}_1, \bar{e}_1)\bar{e}_1)$

$\bar{e}_i := \left[(\dot{\bar{e}}_{i-1})^2 - (\dot{\bar{e}}_{i-1}, \bar{e}_{i-2})^2\right]^{-1/2}\left(\dot{\bar{e}}_{i-1} - (\dot{\bar{e}}_{i-1}, \bar{e}_{i-2})\bar{e}_{i-2}\right)$, $i = 3, \ldots, p$

$\bar{a}_\alpha := \left[(\dot{\bar{a}}_{\alpha-1})^2 - (\dot{\bar{a}}_{\alpha-1}, \bar{a}_{\alpha-2})^2\right]^{-1/2}\left(\dot{\bar{a}}_{\alpha-1} - (\dot{\bar{a}}_{\alpha-1}, \bar{a}_{\alpha-2})\bar{a}_{\alpha-2}\right)$, $\alpha = 3, \ldots, q$

$\bar{x}$ *is the only generating line of* Σ *that satisfies*

$$(1) \qquad \begin{aligned} &(\bar{a}_1, d\bar{x}) = 0 \\ &(\bar{e}_r, d\bar{x}) = 0 \qquad \text{for } 1 \le r < p \end{aligned}$$

(T,3) *The equations of the linear orthonormal frame field* $\bar{A}$ *are of the form*

$$(2) \qquad \dot{\bar{e}}_1 = \bar{a}_1 + F_1^2 \bar{e}_2$$

$$(e) \qquad \begin{aligned} &\dot{\bar{e}}_k = F_k^{k-1}\bar{e}_{k-1} + F_k^{k+1}\bar{e}_{k+1} \quad , \quad k = 2, \ldots, p-1 \\ &\dot{\bar{e}}_p = F_p^{p-1}\bar{e}_{p-1} \quad , \quad F_k^{k+1} = -F_{k+1}^k \end{aligned}$$

$$(a) \qquad \begin{aligned} &\dot{\bar{a}}_1 = -\bar{e}_1 + f_1^2 \bar{a}_2 \\ &\dot{\bar{a}}_\beta = f_\beta^{\beta-1}\bar{a}_{\beta-1} + f^{\beta+1}\bar{a}_{\beta+1} \quad , \quad \beta = 2, \ldots, q-1 \\ &\dot{\bar{a}}_q = f_q^{q-1}\bar{a}_{q-1} \quad , \quad f_\beta^{\beta+1} = -f_{\beta+1}^\beta \end{aligned}$$

$$(x) \qquad \begin{aligned} &\dot{\bar{x}} = f^\alpha \bar{a}_\alpha + F\bar{e}_p \quad , \quad \alpha = 2, \ldots, q \\ &(\dot{\bar{x}} = F\bar{e}_p \quad \text{when} \quad q = 1) \end{aligned}$$

where the parametrization $\overline{\Sigma}$ *of* $[\Sigma]$ *has the property*

$$(3) \qquad (\dot{\bar{e}}_1, \bar{a}_1) := \left(\frac{d\bar{e}_1}{ds}, \bar{a}_1\right) = 1$$

and

$$\begin{aligned} &F_i^{i+1}(s) > 0 \qquad \text{for } i = 1, \ldots, p-2 \\ &f_\gamma^{\gamma+1}(s) > 0 \qquad \text{for } \gamma = 1, \ldots, q-2 \end{aligned}$$

Definition 1. The vector fields $\bar{e}_i$ and $\bar{a}_\alpha$ in (T,2) are called the i-th directional vector of Σ and the α-th normal directional vector of Σ respectively. The only generating line of Σ that satisfies (1) is called the striction line of Σ or of $[\Sigma]$.

Let $[\Sigma]$ be a curve in $P(p,q)$ given with the aid of an a.o.p.c.

$$(4) \qquad \Sigma : s \to \left[Y(s) : (\lambda^i) \to \bar{x}(s) + \lambda^i \bar{e}_i(s)\right]$$

that satisfies the assumptions of Theorem 1, where, in advance, $\bar{e}_i$ and $\bar{x}$ are the vectors and the striction line determined for Σ by (T,2) and the parametrization Σ of $[\Sigma]$ has the property (3).

Let us consider the sequence of curves $\{[\Sigma_k]\}$ given by the sequence of p.c. $\{\Sigma_k\}$, $(k = 0, \ldots, p)$ defined as follows

$$(5) \qquad \begin{aligned} &\text{a)} \quad \Sigma_k : s \to \left[Y_k(s) : (\lambda^{k+1}, \ldots, \lambda^p) \to \bar{x}(s) + \lambda^{k+i}\bar{e}_{k+i}(s)\right] \\ &\qquad \text{when } k = 0, \ldots, p-1 \qquad (\Sigma_k \text{ is a p.c. in } P(p-k, q+k)) \\ &\text{b)} \quad \Sigma_p : s \to \bar{x}(s) \\ &\qquad \text{when } k = p \qquad (\Sigma_p \text{ is a curve in } E^{p+q}) \end{aligned}$$

From the equations (2)(e) it follows that

(6) $$\dim(\operatorname{lin}(\bar{e}_{k+1},\ldots,\bar{e}_p,\dot{\bar{e}}_{k+1},\ldots,\dot{\bar{e}}_p)) = p-k+1$$

i.e. Σ_k is admissible for $k = 0,\ldots,p-1$.
Moreover

(7) $$\dim(\operatorname{lin}(\bar{e}_{k+1},\ldots,\bar{e}_p,\dot{\bar{e}}_{k+2},\ldots,\dot{\bar{e}}_p)) = p-k$$

so

(8) $$\bar{e}_{k+1} := e_*^k$$

is the 1–th directional vector of Σ_k , $(k = 0,\ldots,p-1)$. It can be assumed that $\bar{e}_{k+1}$ is the 1–th orienting directional vector of Σ_k. From (2)(e) and (2)(a) it follows that the 1–th normal directional orienting vector a_*^k of Σ_k is of the form

(9) $$a_*^k = \begin{cases} \bar{e}_k , & k = 1,\ldots,p-1 \\ \bar{a}_1 , & k = 0 . \end{cases}$$

The formulas (T,2) for Σ_k take a form

(10) $$\begin{aligned} &\bar{e}_1^k = e_*^k = \bar{e}_{k+1} \\ &\bar{a}_1^k = a_*^k = \begin{cases} \bar{e}_k , & k = 1,\ldots,p-1 \\ \bar{a}_1 , & k = 0 \end{cases} \\ &\bar{e}_2^k = \bar{e}_{k+2} \\ &\bar{e}_j^k = \bar{e}_{k+j} , \quad j = 3,\ldots,p-k \\ &\bar{a}_2^k = \begin{cases} \bar{a}_2 & , k = 0 \\ \bar{a}_1 & , k = 1 \\ \bar{e}_{k-1} & , k = 2,\ldots,p-1 \end{cases} \end{aligned}$$

$$\bar{a}_\beta^k = \begin{cases} \bar{a}_{\beta-k} & , \beta > k \\ \bar{e}_{k-\beta+1} & , \beta \le k \end{cases} , \qquad \beta = 3,\ldots,q+k$$

From the above formulas it follows that each p.c. Σ_k , $k = 0,\ldots,p-1$, is neither r–cylindric nor orthogonally s–cylindric for any r and s.

Then we have obtained

Theorem 2. *If Σ is an a.o.p.c. in $P(p,q)$ that satisfies the assumptions of Theorem 1 and $\bar{e}_1,\ldots,\bar{e}_p$ are succeeding directional vectors of Σ and $\bar{a}_1,\ldots,\bar{a}_q$ are succeeding normal directional vectors of Σ then each p.c. Σ_k is also an a.o.p.c. that satisfies the assumptions of Theorem 1 and $\bar{e}_{k+1},\ldots,\bar{e}_p$ are succeeding directional vectors of Σ_k and $\bar{e}_k,\bar{e}_{k-1},\ldots,\bar{e}_1,\bar{a}_1,\ldots,\bar{a}_q$ are succeeding normal directional vectors of Σ_k .*

We shall prove a certain property of the sequence $\{[\Sigma_k]\}$, that will explain a geometrical sense of the striction line of $[\Sigma]$.

Let us introduce the following notation. If v is a vector in E^{p+q}, then

$$v^{ok(s)} := v - \sum_{i_k=k+1}^{p} \big(v, \bar{e}_{i_k}(s)\big)\bar{e}_{i_k}(s) \tag{11}$$

It is easy to verify that

$$\lim_{h\to 0}\big(v(s+h)^{ok(s+h)}\big) = v(s)^{ok(s)}\,, \quad \text{where } s \to v(s) \text{ is a vector field} \tag{12}$$

$$\big(v^{ok(s)}, w\big) = \big(v^{ok(s)}, w^{ok(s)}\big) \tag{13}$$

Let $\Sigma_k : s \to \big[Y_k(s)\big]$ be the k-th a.o.p.c. of the sequence $\{\Sigma_k\}$.
Let

$$y_0 = x(s) + \sum_{i_k=k+1}^{p} \lambda^{i_k}\bar{e}_{i_k}(s) \tag{14}$$

be an arbitrary point of the plane $\big[Y_k(s)\big]$ for which there exists a point $y_1 \in \big[Y_k(s+h)\big]$, such that

$$\big|y_1 - y_0\big| = \operatorname{dist}\big(\big[Y_k(s)\big], \big[Y_k(s+h)\big]\big)\,.$$

It is obvious that the vector $y_1 - y_0$ is perpendicular to both of $\big[Y_k(s)\big]$ and $\big[Y_k(s+h)\big]$. Then for any point

$$y_2 = x(s+h) + \sum_{i_k} \mu^{i_k}\bar{e}_{i_k}(s+h) \in \big[Y_k(s+h)\big] \tag{15}$$

the vector $(y_2 - y_0)^{ok(s+h)}$ is perpendicular to both of $\big[Y_k(s)\big]$ and $\big[Y_k(s+h)\big]$, i.e.

$$\big((y_2 - y_0)^{ok(s+h)}, \bar{e}_{j_k}(s)\big) = 0 \tag{16}$$

and

$$\big((y_2 - y_0)^{ok(s+h)}, \bar{e}_{j_k}(s+h)\big) = 0 \qquad \text{for } j_k = k+1, \ldots, p \tag{17}$$

So we have

$$\big((y_2 - y_0)^{ok(s+h)}, \bar{e}_{j_k}(s+h) - \bar{e}_{j_k}(s)\big) = 0 \tag{18}$$

The last equality can be written down in the form

$$\Big(\Big(x(s+h) - x(s) + \sum_{i_k} \lambda^{i_k}\big(\bar{e}_{i_k}(s+h) - \bar{e}_{i_k}(s)\big)\Big)^{ok(s+h)}, \bar{e}_{j_k}(s+h) - \bar{e}_{j_k}(s)\Big) = 0 \tag{19}$$

We have obtained the system of equations on λ^{i_k}, $i_k = k+1,\dots,p$. The solutions of (19) determine by (14) the admissible positions of y_0 on $[Y_k(s)]$ for fixed $[Y_k(s+h)]$. Let us compute the values of λ^{i_k} when $h \to 0$. We have

$$\text{(20)} \quad \begin{aligned} &\lim_{h\to 0}\Bigg(\Big(\frac{1}{h}(x(s+h)-x(s)) + \sum_{i_k}\lambda^{i_k}\frac{1}{h}(e_{i_k}(s+h)-e_{i_k}(s))\Big)^{ok(s+h)}, \\ &\frac{1}{h}(e_{j_k}(s+h)-e_{j_k}(s))\Bigg) = 0. \end{aligned}$$

In virtue of (12), (20) takes a form

$$\text{(21)} \qquad \Big(\big(\dot{x}(s) + \sum_{i_k}\lambda^{i_k}\dot{e}_{i_k}(s)\big)^{ok(s)}, \dot{e}_{j_k}(s)\Big) = 0$$

Then in virtue of (13), (21) takes a form

$$\text{(22)} \quad (\dot{x}^{ok}, \dot{e}_{j_k}{}^{ok}) + \sum_{i_k}\lambda^{i_k}(\dot{e}_{i_k}{}^{ok}, \dot{e}_{j_k}{}^{ok}) = 0 \quad , \quad k = 0,\dots,p-1\ ,\ j_k = k+1,\dots,p.$$

Using (2)(e)(a)(x) we compute coefficients of the above system of equations, and obtain

$$\text{(23)} \qquad \begin{cases} \lambda^{k+1}G_{k+1} = 0 & \text{(the 1–th equation, i.e. } j_k = k+1\text{)} \\ \qquad\ \ 0 = 0 & \text{(the next equations, i.e. } j_k > k+1\text{)} \end{cases}$$

where

$$G_{k+1} = \begin{cases} 1 & , k = 0 \\ (F^k_{k+1})^2 & , k > 0 \end{cases}$$

The solution of (23) is of the form

$$\text{(24)} \qquad \lambda^{k+1} = 0\,, \qquad \lambda^{k+2},\dots,\lambda^p - \text{arbitrary}\,,$$

so the set of all considered points $y_0 \in [Y_k(s)]$ coincides with the plane $[Y_{k+1}(s)]$.

We have obtained the following theorem that gives the geometrical inerpretation of the striction line.

Theorem 3. *Given an a.o.p.c. Σ, that satisfies the assumptions of Theorem 1. The set of all points y_0 of the plane $[Y_k(s)]$ of the a.o.p.c. Σ_k $k = 0,\dots,p-1$, for which there exists a point y_1 of the approaching plane $[Y_k(s+h)]$ $(h \to 0)$ such that $|y_0 - y_1| = dist\,([Y_k(s)],[Y_k(s+h)])$*

a) *coincides with the plane $[Y_{k+1}(s)]$ of the a.o.p.c. Σ_{k+1} when $k < p-1$*

b) *contains exactly one point and it is a point $x(s)$ of the striction line of $[\Sigma]$ when $k = p-1$.*

The above theorem explains, that a notion of the striction line of a curve in $P(p,q)$ is the simple generalization of a notion of the striction line of a ruled surface in E^3 understood as a curve in $P(1,2)$. (see [6] or [4]). The following corollary is a consequence of Theorem 3.

Corollary . The striction line of a curve $[\Sigma]$ is simultaneously the striction line of each curve $[\Sigma_k]$ $(k = 0, \ldots, p-1)$ of the sequence defined for $[\Sigma]$ by (5).

A curve in $P(p,q)$ can be considered as a surface in E^{p+q}. Let us introduce the following definition.

Definition 2. A surface $S[\Sigma]$ of a curve $[\Sigma]$ in $P(p,q)$ is a set in E^{p+q} that consists of all points of all planes of $[\Sigma]$.

When $\Sigma : t \to [Y(t) : (\lambda^i) \to x(t) + \lambda^i e_i(t)]$ is a p.c. in $P(p,q)$ then

$$\begin{gathered} S\Sigma : R^{p+1} \longrightarrow E^{p+q} \\ (t, \lambda^1, \ldots, \lambda^p) \to y(t, \lambda^1, \ldots, \lambda^p) = x(t) + \lambda^i e_i(t) \end{gathered} \tag{24}$$

is a parametrization of the surface $S[\Sigma]$ of the curve $[\Sigma]$. We shall prove the following theorem.

Theorem 4. *The surface $S[\Sigma_{k+1}]$ of the curve $[\Sigma_{k+1}]$ of the sequence (5) is the set of singularities of the surface $S[\Sigma_k]$ of the curve $[\Sigma_k]$ for $k = 0, \ldots, p-2$. The striction line $\bar{x}$ of Σ is the set of singularities of the surface $S[\Sigma_{p-1}]$.*

Proof. Let us consider the parametrization

$$S\Sigma_k : (s, \lambda^{k+1}, \ldots, \lambda^p) \to y(s, \lambda^{k+1}, \ldots, \lambda^p) = \bar{x}(s) + \sum_{i_k = k+1}^{p} \lambda^{i_k} \bar{e}_{i_k}(s)$$

of $S[\Sigma_k]$. A tangent space of $S[\Sigma_k]$ is spanned by vectors

$$\begin{aligned} \frac{\partial S\Sigma_k}{\partial s} &= \dot{\bar{x}} + \sum_{i_k = k+1}^{p} \lambda^{i_k} \dot{\bar{e}}_{i_k} \\ \frac{\partial S\Sigma_k}{\partial \lambda^{j_k}} &= \bar{e}_{j_k} \quad , \quad j_k = k+1, \ldots, p \, . \end{aligned} \tag{25}$$

Then

$$\dot{\bar{x}}^{ok} + \sum_{i_k} \lambda^{i_k} (\dot{\bar{e}}_{i_k})^{ok} = 0 \qquad (\text{ see } (11)) \tag{26}$$

is an equation on singularities of $S[\Sigma_k]$. The above vectorial equation is equivalent to the following system of scalar equations

$$(\dot{\bar{x}}^{ok}, \dot{\bar{e}}_{j_k}{}^{ok}) + \sum_{i_k} \lambda^{i_k} (\dot{\bar{e}}_{i_k}{}^{ok}, \dot{\bar{e}}_{j_k}{}^{ok}) = 0 \tag{27}$$

that is similar to (22).

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STRESZCZENIE

W pracy [5] skonstruowano pole ortonormalnych reperów liniowych wzdłuż krzywej w przestrzeni $P(p,q)$ płaszczyzn p-wymiarowych w $(p+q)$-wymiarowej przestrzeni euklidesowej. W niniejszej pracy przedstawiamy pewne własności wektorów otrzymanego pola reperów i wyjaśniamy sens geometryczny pojęcia linii strykcji krzywej w $P(p,q)$.

SUMMARY

In [5] the linear orthonormal frame field along a curve in the space $P(p,q)$ of p-dimensional planes in the $(p+q)$-dimensional euclidean space has been constructed. In this paper some properties of vector fields of the constructed frame field are presented and a geometrical interpretation of the striction line of a curve in $P(p,q)$ is given.