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π-Projective Curvature Tensors

π-projektywne tensory krzywizny

1. Introduction. Let V_n^Γ be an n-dimensional space with a linear connection Γ given with the aid of components Γ^i_{jk} in each local map U on a differentiable manifold V_n. Let us consider a fixed differentiable tensor field π of the type $(0,2)$ on V_n. It will be called non-singular if there exists an atlas A on V_n such that $\det(\pi_{ij}) \neq 0$ in each local cart of A.

K. Radziszewski introduced [7] following notions:

A vector field w^i is called π-geodesic if

$$\nabla_k(\pi_{ia}w^a)w^k = \lambda \pi_{ia}w^a$$

where ∇_k denotes the covariant derivative with respect to the connection Γ.

Integral curve of π-geodesic vector fields is called π-geodesic.

K.Radziszewski obtained the differential equations of π-geodesics in the form

$$\frac{d^2x^i}{dt^2} + \left(\nabla_k\pi_{ps}\bar{\pi}^{pi} + \Gamma^i_{ks}\right)\frac{dx^k}{dt}\frac{dx^s}{dt} = \lambda\frac{dx^i}{dt}$$

where $\lambda = \lambda(x^j)$, and $\bar{\pi}^{pi}$ is defined by $\pi_{ps}\bar{\pi}^{pi} = \delta^i_s$. These equations show that π-geodesics in a space V_n^Γ are geodesics in ordinary sence in the space V_n^G, where connection G is given by

$$G^i_{ks} = \Gamma^i_{ks} + (\nabla_k\pi_{ps})\bar{\pi}^{pi} \, . \tag{1.1}$$

In [7] some theorems concerning tensor π or tensors π and $\bar{\pi}$ are proved such that π-geodesics be geodesics in ordinary sence or such that π- geodesics and $\bar{\pi}$-geodesics are the same curves in V_n. A. Bucki [2] considered π- geodesics with respect to the third and fourth fundamental tensors of hypersurfaces.

The object of the present paper is the investigation of the π-projective transformations.

In §2 we find a change of Γ^i_{jk} which does not change the system of π- geodesics and define π-projective transformation of Γ.

Supposing that the connection Γ is symmetric and that the tensor π is symmetric and satisfies the condition (2.4), we find in §3 the first and in §4 the second π-projective curvature tensor. In §5 we discuss the case of Riemannian space whose π-projective curvature tensor vanishes.

2. π-projective transformations. We have to find a change of Γ which does not change the system of π-geodesics. We have seen in §1 that π-geodesic is a geodesic in ordinary sence with respect to the connection (1.1). Assuming that G^i_{ks} is a symmetric connection, a change of G^i_{ks} which does not change the system of geodesics is given by

$$\bar{\Gamma}^i_{ks} = G^i_{ks} + \delta^i_k \psi_s + \delta^i_s \psi_k \, , \tag{2.1}$$

where ψ_s is an arbitrary covector field. As for $\bar{\Gamma}^i_{ks}$ it is also symmetric connection and must determine π-geodesics too, i.e. it must have the form

$$\bar{\Gamma}^i_{ks} = \bar{\Gamma}^i_{ks} + (\overline{\nabla}_k \pi_{ps}) \tilde{\pi}^{pi} \, . \tag{2.2}$$

We have from (2.2) and (1.1)

$$\bar{\Gamma}^i_{ks} = \frac{\partial \pi_{ps}}{\partial x^k} \tilde{\pi}^{pi} = \overline{\Gamma}^a_{kp} \pi_{as} \tilde{\pi}^{pi}$$
$$G^i_{ks} = \frac{\partial \pi_{ps}}{\partial x^k} \tilde{\pi}^{pi} = \Gamma^a_{kp} \pi_{as} \tilde{\pi}^{pi} \, .$$

Therefore

$$\bar{\Gamma}^i_{ks} - G^i_{ks} = -(\overline{\Gamma}^a_{kp} - \Gamma^a_{kp}) \pi_{as} \tilde{\pi}^{pi} \, .$$

Transvecting this equation with $\pi_{ij} \tilde{\pi}^{st}$, we find

$$-\overline{\Gamma}^t_{kj} + \Gamma^t_{kj} = (\bar{\Gamma}^i_{ks} - G^i_{ks}) \pi_{ij} \tilde{\pi}^{st}$$

from which, using (2.1), we obtain

$$\overline{\Gamma}^t_{kj} = \Gamma^t_{kj} - \pi_{kj} \psi_s \tilde{\pi}^{st} - \psi_k \delta^t_j \, . \tag{2.3}$$

Conversely, let us suppose that (2.3) holds good. We then can express the connection

$$\bar{\Gamma}^i_{kt} = \Gamma^i_{kt} + (\nabla_k \pi_{pt}) \tilde{\pi}^{pi}$$

in the form

$$\bar{\Gamma}^i_{kt} = \Gamma^i_{kt} + (\nabla_k \pi_{pt}) \tilde{\pi}^{pi} + \psi_t \delta^i_k + \psi_k \delta^i_t \, .$$

This shows that the space V^G_n and $V^{\bar{\Gamma}}_n$ where

$$\bar{\Gamma}^i_{kt} = \overline{\Gamma}^i_{kt} + (\overline{\nabla}_k \pi_{pt}) \tilde{\pi}^{pi} \quad \text{and} \quad G^i_{kt} = \Gamma^i_{kt} + (\nabla_k \pi_{pt}) \tilde{\pi}^{pi}$$

have the same geodesics, or, equivalently, that the spaces V^{Γ}_n and $V^{\overline{\Gamma}}_n$ have the same π-geodesics.

Thus we have

Theorem. *Condition* (2.3) *where ψ_k is an arbitrary covector field, is necessary and sufficient for the spaces V_n^{Γ} and $V_n^{\overline{\Gamma}}$ to have the same π-geodesics.*

Definition. A change (2.3) is called π– projective transformation.

In the above consideration we have supposed that both connections G and $\overline{\Gamma}$ are symmetric. In the suite, we supoose that the connection Γ^i_{jk} and the tensor field π_{ij} are symmetric too. The symmetry of the connections Γ and G implies the following condition

$$\nabla_k \pi_{ij} - \nabla_j \pi_{ik} = 0 . \tag{2.4}$$

3. First π–projective curvature tensor. Let us compute the curvature tensor of $\overline{\Gamma}^i_{ks}$:

$$\overline{R}^i_{hkj} = \frac{\partial \overline{\Gamma}^i_{jh}}{\partial x^k} - \frac{\partial \overline{\Gamma}^i_{kh}}{\partial x^j} + \overline{\Gamma}^i_{kq}\,\overline{\Gamma}^q_{jp} - \overline{\Gamma}^q_{jq}\,\overline{\Gamma}^q_{kp} .$$

By straightforward computation we find

$$\begin{aligned} \overline{R}^i_{hkj} = R^i_{hkj} &+ (\nabla_j \pi_{kh} - \nabla_k \pi_{jh})\bar{\pi}^{si}\psi_s + \delta^i_h(\nabla_j\psi_k - \nabla_k\psi_j) + \\ &+ \pi_{jh}\left[\bar{\pi}^{ai}\psi_k\psi_a - \nabla_k(\psi_s\bar{\pi}^{si})\right] - \pi_{kh}\left[\bar{\pi}^{ai}\psi_j\psi_a - \nabla_j(\psi_s\bar{\pi}^{si})\right] , \end{aligned} \tag{3.1}$$

where R^i_{hkj} are the components of the curvature tensor of the connection Γ^i_{jk}. Taking account of (2.4), we rewrite (3.1) in the form

$$\overline{R}^i_{hkj} = R^i_{hkj} + \delta^i_h \varphi_{jk} + \pi_{jh}\theta^i_k - \pi_{kh}\theta^i_j , \tag{3.2}$$

where we have put

$$\varphi_{jk} = \nabla_j\psi_k - \nabla_k\psi_j \quad , \quad \theta^i_k = \bar{\pi}^{ai}\psi_k\psi_a - \nabla_k(\psi_s\bar{\pi}^{si}) \tag{3.3.}$$

Contracting (3.2) with respect to i and r we get

$$\overline{R}^a_{akj} = R^a_{akj} + n\,\varphi_{jk} + \pi_{ja}\theta^a_k - \pi_{ka}\theta^a_j . \tag{3.4}$$

On the other hand, from

$$\bar{\pi}^{sa}\pi_{ja} = \delta^s_j$$

we have

$$-(\nabla_k\bar{\pi}^{sa})\pi_{ja} = \bar{\pi}^{sa}(\nabla_k\pi_{ja}) .$$

We then obtain

$$\pi_{ja}\theta^a_k = \psi_k\psi_j - \nabla_k\psi_j - \psi_s(\nabla_k\bar{\pi}^{sa})\pi_{ja} = \psi_k\psi_j - \nabla_k\psi_j + \psi_s(\nabla_k\pi_{ja})\bar{\pi}^{sa} .$$

Consequently

$$\pi_{ja}\theta^a_k - \pi_{ka}\theta^a_j = \nabla_j\psi_k - \nabla_k\psi_j + \psi_s\bar{\pi}^{sa}(\nabla_k\pi_{ja} - \nabla_j\pi_{ka})$$

or using (2.4) and (3.3)

$$\pi_{ja}\theta^a_k - \pi_{ka}\theta^a_j = \varphi_{jk} \, .$$

Substituting this into (3.4), we find

$$\varphi_{jk} = \frac{1}{n+1}(\overline{R}^a_{akj} - R^a_{akj}) \, . \tag{3.5}$$

Contracting (3.2) with respect to i and j, we get

$$\overline{R}_{hk} = R_{hk} + \varphi_{hk} + \pi_{ah}\theta^a_k - \pi_{kh}\theta^a_a \, . \tag{3.6}$$

where

$$\overline{R}_{hk} = \overline{R}^a_{hka} \quad , \qquad R_{hk} = R^a_{hka} \, .$$

Tensor φ_{hk} being skew–symmetric and tensor $\tilde{\pi}^{hk}$ symmetric, we have $\varphi_{hk}\tilde{\pi}^{hk} = 0$. Thus, transvecting (3.6) with $\tilde{\pi}^{hk}$, we find

$$\theta^a_a = \frac{1}{n-1}(\tilde{\pi}^{ab}\overline{R}_{ab} - \tilde{\pi}^{ab}R_{ab}) \, .$$

Substituting this and (3.5) into (3.6) and then transvecting with $\tilde{\pi}^{ih}$ we obtain

$$\theta^i_k = \tilde{\pi}^{ia}\overline{R}_{ak} - \frac{\delta^i_k}{n-1}\tilde{\pi}^{ab}\overline{R}_{ab} - \tilde{\pi}^{ia}R_{ak} + \frac{\delta^i_k}{n-1}\tilde{\pi}^{ab}R_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}(\overline{R}^a_{abk} - R^a_{abk}). \tag{3.7}$$

Taking account of (3.7) and (3.5), we can write (3.2) in the form

$$\begin{aligned}
\overline{R}^i_{hkj} - \frac{1}{n+1}\delta^i_h\overline{R}^a_{akj} - \pi_{jh}\Big(\tilde{\pi}^{ia}\overline{R}_{ak} - \frac{1}{n-1}\delta^i_k\tilde{\pi}^{ab}\overline{R}_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}\overline{R}^a_{abk}\Big) + \\
+ \pi_{kh}\Big(\tilde{\pi}^{ia}\overline{R}_{aj} - \frac{1}{n-1}\delta^i_j\tilde{\pi}^{ab}\overline{R}_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}\overline{R}^a_{abj}\Big) = \\
= R^i_{hkj} - \frac{1}{n+1}\delta^i_h R^a_{akj} - \pi_{jh}\Big(\tilde{\pi}^{ia}R_{ak} - \frac{1}{n-1}\delta^i_k\tilde{\pi}^{ab}R_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}\overline{R}^a_{abk}\Big) + \\
+ \pi_{kh}\Big(\tilde{\pi}^{ia}R_{aj} - \frac{1}{n-1}\delta^i_j\tilde{\pi}^{ab}R_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}R^a_{abj}\Big) \, .
\end{aligned}$$

Therefore, the tensor

$$\begin{aligned}
\underset{1}{P}{}^i_{hkj} = R^i_{hkj} - \frac{1}{n+1}\delta^i_h R^a_{akj} - \pi_{jh}\Big(\tilde{\pi}^{ia}R_{ak} - \frac{1}{n-1}\delta^i_k\tilde{\pi}^{ab}R_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}R^a_{abk}\Big) + \\
+ \pi_{kh}\Big(\tilde{\pi}^{ia}R_{aj} - \frac{1}{n-1}\delta^i_j\tilde{\pi}^{ab}R_{ab} - \frac{1}{n+1}\tilde{\pi}^{ib}R^a_{abj}\Big)
\end{aligned} \tag{3.8}$$

is invariant under a π–projective change (2.3).

We call the tensor (3.8) the first π–projective curvature tensor.

4. The second π–projective curvature tensor. Although condition Γ^i_{jk} is symmetric, connection (2.3) is not. Denoting by $\underset{1}{\overline{\nabla}}$ the operator of covariant differentiation with respect to $\overline{\Gamma}^i_{jk}$, we define the second one by the equation

$$\underset{2}{\overline{\nabla}}_X Y - \underset{1}{\overline{\nabla}}_Y X = [X, Y] \, ,$$

where X and Y are arbitrary vector fields. Then we can prove ([5], [6]) the existence of four curvature tensors:

$$\underset{1}{\overline{R}}(X,Y)Z = \underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{1}{\nabla}_X Z - \underset{1}{\nabla}_{[X,Y]} Z \ ,$$

$$\underset{2}{\overline{R}}(X,Y)Z = \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{2}{\nabla}_X Z - \underset{2}{\nabla}_{[X,Y]} Z \ ,$$

$$\underset{3}{\overline{R}}(X,Y)Z = \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z \ ,$$

$$\underset{4}{\overline{R}}(X,Y)Z = \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z \ ,$$

In the coordinate system of a local map of V_n, these tensors have the components as follows:

$$\underset{1}{\overline{R}}{}^i_{hkj} = \frac{\partial \overline{\Gamma}^i_{hj}}{\partial x^k} - \frac{\partial \overline{\Gamma}^i_{hk}}{\partial x^j} + \overline{\Gamma}^i_{qk}\overline{\Gamma}^q_{hj} - \overline{\Gamma}^i_{qj}\overline{\Gamma}^q_{hk} \ ,$$

$$\underset{2}{\overline{R}}{}^i_{hkj} = \frac{\partial \overline{\Gamma}^i_{jh}}{\partial x^k} - \frac{\partial \overline{\Gamma}^i_{kh}}{\partial x^j} + \overline{\Gamma}^i_{kq}\overline{\Gamma}^q_{jh} - \overline{\Gamma}^i_{jq}\overline{\Gamma}^q_{kh} \ ,$$

$$\underset{3}{\overline{R}}{}^i_{hkj} = \frac{\partial \overline{\Gamma}^i_{hj}}{\partial x^k} - \frac{\partial \overline{\Gamma}^i_{kh}}{\partial x^j} + \overline{\Gamma}^i_{ka}\overline{\Gamma}^a_{hj} - \overline{\Gamma}^i_{aj}\overline{\Gamma}^a_{kh} + \overline{\Gamma}^a_{kj}(\overline{\Gamma}^i_{ah} - \overline{\Gamma}^i_{ha}) \ ,$$

$$\underset{4}{\overline{R}}{}^i_{hkj} = \frac{\partial \overline{\Gamma}^i_{hj}}{\partial x^k} - \frac{\partial \overline{\Gamma}^i_{kh}}{\partial x^j} + \overline{\Gamma}^i_{ka}\overline{\Gamma}^a_{hj} - \overline{\Gamma}^i_{aj}\overline{\Gamma}^a_{kh} + \overline{\Gamma}^a_{jk}(\overline{\Gamma}^i_{ah} - \overline{\Gamma}^i_{ha}) \ .$$

The tensor $\overline{R}^i_{hkj}$ appearing in §2 is, in fact, tensor $\underset{2}{\overline{R}}_{hkj}$. We shall show in this paragraph that we can construct, using the tensor $\underset{3}{\overline{R}}$ the new π–projective curvature tensor.

In fact, taking account of (2.4) we can easily verify that

$$\underset{3}{\overline{R}}{}^i_{hkj} = R^i_{hkj} + \pi_{kh}\left[\nabla_j(\bar{\pi}^{ai}\psi_a) - \bar{\pi}^{ai}\psi_a\psi_j\right] - \pi_{jh}\left[\nabla_k(\bar{\pi}^{ai}\psi_a) - \bar{\pi}^{ai}\psi_a\psi_k\right] +$$
$$+ \delta^i_h\left(\nabla_j\psi_k + \psi_k\psi_j + \pi_{kj}\bar{\pi}^{ap}\psi_a\psi_p\right) - \delta^i_j\left(\nabla_k\psi_j + \psi_k\psi_j + \pi_{kh}\bar{\pi}^{ap}\psi_a\psi_p\right),$$

or

$$\underset{3}{\overline{R}}{}^i_{hkj} = R^i_{hkj} - \pi_{kh}\theta^i_j + \pi_{jh}\theta^i_k + \delta^i_h\beta_{jk} - \delta^i_j\beta_{kh} \ , \tag{4.1}$$

where, as in §3, we have put

$$\theta^i_j = \bar{\pi}^{ia}\psi_j\psi_a - \nabla_j(\psi_a\bar{\pi}^{ai}) \ ,$$

R^i_{hkj} are the components of the curvature tensor of the connection Γ^i_{jk} , and

$$\beta_{jh} = \nabla_j\psi_h + \psi_h\psi_j + \pi_{jh}\bar{\pi}^{pa}\psi_p\psi_a \ . \tag{4.2}$$

Contracting (4.1) with respect to i and j, we have

$$\underset{3}{\overline{R}}{}^a_{hka} = R_{hk} - \pi_{kh}\theta^a_a + \pi_{ha}\theta^a_k + \beta_{hk} - n\,\beta_{kh} \ . \tag{4.3}$$

On the other hand, contracting (4.1) with respect to i and r, we find

$$\underset{3}{\overline{R}}{}^{a}_{akj} = R^{a}_{akj} - \pi_{ha}\theta^{a}_{j} + \pi_{aj}\theta^{a}_{k} + n\,\beta_{jk} - \beta_{kj}\ . \tag{4.4}$$

Transvecting (4.3) with $\tilde{\pi}^{hk}$ and (4.4) with $\tilde{\pi}^{kj}$, we get

$$\tilde{\pi}^{bc}\underset{3}{R}{}^{a}_{bca} = \tilde{\pi}^{bc}R_{bc} - (n-1)\theta^{a}_{a} - (n-1)\beta_{ab}\tilde{\pi}^{ab}\ ,$$

$$\tilde{\pi}^{bc}\underset{3}{\overline{R}}{}^{a}_{abc} = (n-1)\beta_{ab}\tilde{\pi}^{ab}\ .$$

Adding the last two equations, we obtain

$$\theta^{a}_{a} = -\underset{3}{\overline{R}}{}' + R'\ ,$$

where we have put

$$\begin{cases} \underset{3}{\overline{R}}{}' = \dfrac{1}{(n-1)}\tilde{\pi}^{st}\left(\underset{3}{\overline{R}}{}^{a}_{sta} + \underset{3}{\overline{R}}{}^{a}_{ast}\right) \\ R' = \dfrac{1}{(n-1)}\tilde{\pi}^{st}R_{st}\ . \end{cases} \tag{4.5}$$

Substituting θ^{a}_{a} into (4.3), we have

$$\underset{3}{\overline{R}}{}^{a}_{hka} - R_{hk} + \pi_{kh}\left(-\underset{3}{\overline{R}}{}' + R'\right) = \pi_{ha}\theta^{a}_{k} + \beta_{hk} - n\,\beta_{kh}\ . \tag{4.6}$$

On the other hand, transvecting (4.1) with $\tilde{\pi}^{kj}$ and taking into account that the tensor R^{i}_{hkj} is skew-symmetric with respect to k and j, we find

$$\underset{3}{\overline{R}}{}^{i}_{hab}\tilde{\pi}^{ab} = \delta^{i}_{h}\ \beta_{ab}\ \tilde{\pi}^{ab} - \tilde{\pi}^{ia}\beta_{ah}\ , \tag{4.7}$$

from which, contracting with respect to i and r, we get

$$\tilde{\pi}^{ab}\beta_{ab} = \frac{1}{(n-1)}\underset{3}{\overline{R}}{}^{d}_{dab}\tilde{\pi}^{ab}\ .$$

Substituting this into (4.7), we obtain

$$\tilde{\pi}^{ia}\beta_{ah} = -\underset{3}{\overline{R}}{}^{i}_{hab}\tilde{\pi}^{ab} + \frac{\delta^{i}_{h}}{n-1}\underset{3}{\overline{R}}{}^{d}_{dab}\tilde{\pi}^{ab}$$

or

$$\beta_{jh} = -\pi_{dj}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^{d}_{hab} + \frac{1}{n-1}\pi_{jh}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^{d}_{dab}\ . \tag{4.8}$$

Substituting into (4.6), we find

$$\pi_{ha}\theta^{a}_{k} = \underset{3}{\overline{R}}{}^{a}_{hka} + \pi_{hd}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^{d}_{kab} - n\ \pi_{kd}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^{d}_{hab} - \underset{3}{\overline{R}}\pi_{kh} - \left(R_{hk} - R'\pi_{hk}\right)$$

where we have put

$$\underset{3}{\overline{R}} = \underset{3}{\overline{R}'} - \tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^d_{dab} .$$

Transvecting the preceding equation with $\tilde{\pi}^{hi}$, we obtain

$$\text{(4.9)} \qquad \theta^i_k = \tilde{\pi}^{bi}\underset{3}{\overline{R}}{}^a_{bka} + \tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^i_{kab} - n\,\pi_{kd}\tilde{\pi}^{ab}\tilde{\pi}^{ti}\underset{3}{\overline{R}}{}^d_{tab} - \underset{3}{\overline{R}}\delta^i_k - \left(\tilde{\pi}^{ia}R_{ak} - R'\delta^i_k\right) .$$

Substituting (4.8) and (4.9) into (4.1), we find

$$\begin{aligned} &\underset{3}{\overline{R}}{}^i_{hkj} + \pi_{kh}\left(\tilde{\pi}^{bi}\underset{3}{\overline{R}}{}^a_{bja} + \tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^i_{jab} - n\,\pi_{jd}\tilde{\pi}^{ab}\tilde{\pi}^{ti}\underset{3}{\overline{R}}{}^d_{tab} - \underset{3}{\overline{R}}\delta^i_j\right) - \\ &\quad - \pi_{jh}\left(\tilde{\pi}^{bi}\underset{3}{\overline{R}}{}^a_{bka} + \tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^i_{kab} - n\,\pi_{kd}\tilde{\pi}^{ab}\tilde{\pi}^{ti}\underset{3}{\overline{R}}{}^d_{tab} - \underset{3}{\overline{R}}\delta^i_k\right) - \\ &\quad - \delta^i_h\left(-\pi_{dj}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^d_{kab} + \frac{\pi_{jk}}{n-1}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^d_{dab}\right) + \delta^i_j\left(-\pi_{dk}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^d_{hab} + \frac{\pi_{hk}}{n-1}\tilde{\pi}^{ab}\underset{3}{\overline{R}}{}^d_{dab}\right) = \\ &= R^i_{hkj} + \pi_{kh}\left(\tilde{\pi}^{bi}R_{bj} - R'\delta^i_j\right) - \pi_{jh}\left(\tilde{\pi}^{bi}R_{bk} - R'\delta^i_k\right) . \end{aligned}$$

Thus, the tensor on the left hand side is invariant under π–projective change (2.3). We call it the second π–projective curvature tensor and denote it by $\underset{2}{P}{}^i_{hkj}$. Therefore, we have

$$\text{(4.10)} \qquad \underset{2}{P}{}^i_{hkj} = R^i_{hkj} + \pi_{kh}\left(\tilde{\pi}^{bi}R_{bj} - R'\delta^i_j\right) - \pi_{jh}\left(\tilde{\pi}^{bi}R_{bk} - R'\delta^i_k\right) .$$

Remark . We may use, instead of $\underset{2}{\overline{R}}{}^i_{hkj}$ and $\underset{3}{\overline{R}}{}^i_{hkj}$ the tensors $\underset{1}{\overline{R}}{}^i_{hkj}$ and $\underset{4}{\overline{R}}{}^i_{hkj}$. In that case, we have to use instead of operator $\underset{2}{\overline{\nabla}}$ the operator $\underset{1}{\overline{\nabla}}$ and proceeding in a similar manner as in §2, we find

$$\overline{\Gamma}^i_{kj} = \Gamma^i_{kj} - \pi_{kj}\psi_s\tilde{\pi}^{si} - \psi_j\delta^i_k$$

instead of (2.3). By straightforward computation we obtain

$$\underset{1}{\overline{R}}{}^i_{hkj} = R^i_{hkj} + \delta^i_h\left(\nabla_j\psi_k - \nabla_k\psi_j\right) + \pi_{hj}\left[\psi_k\psi_s\tilde{\pi}^{si} - \nabla_k\left(\psi_s\tilde{\pi}^{si}\right)\right] - \pi_{hk}\left[\psi_j\psi_s\tilde{\pi}^{si} - \nabla_j\left(\psi_s\tilde{\pi}^{si}\right)\right].$$

Comparing to (3.1), we see at once that the tensor $\underset{1}{\overline{R}}$ leads to the first π–projective curvature tensor $\underset{1}{P}{}^i_{hkj}$. In the similar manner, starting from $\underset{4}{\overline{R}}$, we get $\underset{2}{P}{}^i_{hkj}$.

5. Riemannian space whose π–projective curvature tensor vanishes. We shall now consider the case when V^Γ_n is a Riemannian space, i.e. when Γ^i_{jk} are Christoffel symbols with respect to metric tensor g_{ij} of the Riemannian space. Denoting by K^i_{hkj} the components of Riemannian curvature tensor, we have

$$R^i_{hkj} = K^i_{hkj} \quad , \quad R^a_{akj} = 0 .$$

Thus we have

$$\underset{1}{P}{}^i_{hkj} = \underset{2}{P}{}^i_{hkj} = P^i_{hkj} = K^i_{hkj} + \pi_{kh}\left(\tilde{\pi}^{bi}K_{bj} - R\delta^i_j\right) - \pi_{jh}\left(\tilde{\pi}^{bi}K_{bk} - R\delta^i_j\right)$$

where we have put

$$R = \frac{1}{n-1} \bar{\pi}^{ab} K_{ab} ,$$

and K_{ij} is the Ricci tensor of a Riemannian space.

It is easily to see that the tensor P^i_{hkj} is skew-symmetric with respect to k and j and satisfies the conditions

$$P^i_{hkj} + P^i_{kjh} + P^i_{jhk} = 0 \quad , \quad P^a_{hka} = 0 .$$

If the Ricci tensor of a Riemannian space satisfies the condition

$$\nabla_k K_{ji} - \nabla_j K_{ki} = 0 , \tag{5.1}$$

then we can choose $\pi_{ij} = K_{ij}$. In that case the tensor P reduces to the form

$$P^i_{hkj} = K^i_{hkj} - \frac{1}{n-1}\left(\delta^i_j K_{kh} - \delta^i_j K_{jh}\right) .$$

The condition (5.1) being equivalent with the condition

$$\nabla_a K^a_{hkj} = 0 , \tag{5.2}$$

we have

Theorem. *For the Riemannian space satisfying the condition* (5.2) *tensor* P^i_{hkj} *with respect to the Ricci tensor reduces to the ordinary projective curvature tensor.*

Now we suppose that this π-projective curvature tensor vanishes. Then we have

$$K^i_{hkj} = \pi_{hj}\left(\bar{\pi}^{bi} K_{bk} - R\,\delta^i_k\right) - \pi_{kh}\left(\bar{\pi}^{bi} K_{bj} - R\,\delta^i_j\right) ,$$

or, raising the indice i ,

$$K_{ihkj} = \pi_{hj} S_{ik} - \pi_{hk} S_{ij} , \tag{5.3}$$

where we have put

$$S_{ik} = g_{ia} \bar{\pi}^{ba} K_{bk} - R g_{ik} .$$

The tensor K_{ihkj} being skew-symmetric in i and r, we find

$$\pi_{hj} S_{ik} - \pi_{kh} S_{ij} + \pi_{ij} S_{hk} - \pi_{ki} S_{hj} = 0 .$$

Transvecting this with $\bar{\pi}^{hj}$, we get

$$S_{ik} = S \pi_{ik} \qquad \text{where} \quad S = \frac{1}{n} S_{ab} \bar{\pi}^{ah} .$$

This means that (5.3) reduces to

$$K_{ihkj} = S\left(\pi_{hj}\pi_{ik} - \pi_{kh}\pi_{ij}\right) . \tag{5.4}$$

Conversely, if (5.4) is satisfied, we have

$$R = \frac{1}{n-1}\bar{\pi}^{hk}K_{hk} = -S\pi_{pq}g^{pq}$$
$$S_{ik} = Sg_{ia}\bar{\pi}^{ba}\left(g^{pq}\pi_{bq}\pi_{pk} - \pi_{bk}\pi_{pq}g^{pq}\right) + Sg_{ik}\pi_{pq}g^{pq} = S\pi_{ik}\ .$$

Therefore

$$P_{ihkj} = K_{ihkj} + \pi_{kh}S_{ij} - \pi_{jh}S_{ik} = S\left(\pi_{hj}\pi_{ik} - \pi_{hk}\pi_{ij}\right) + S\left(\pi_{kh}\pi_{ij} - \pi_{hj}\pi_{ik}\right) = 0.$$

Thus we have

Theorem . *π-projective curvature tensor of a Riemannian space vanishes if and only if its curvature tensor has the form* (5.4).

It is easy to see that scalar S in (5.4) is a constant. In fact, using the identity of Bianchi and (2.4), we find

$$\frac{\partial S}{\partial x^l}\left(\pi_{hk}\pi_{ij} - \pi_{hj}\pi_{ik}\right) + \frac{\partial S}{\partial x^k}\left(\pi_{jh}\pi_{il} - \pi_{lh}\pi_{ij}\right) + \frac{\partial S}{\partial x^j}\left(\pi_{lh}\pi_{ik} - \pi_{kh}\pi_{il}\right) = 0\ .$$

Transvecting with $\bar{\pi}^{kh}$ and $\bar{\pi}^{ij}$ and supposing $n > 2$, we get $\dfrac{\partial S}{\partial x^l} = 0$.

Consequently we have

Theorem . *If π-projective curvature tensor of a Riemannian space V_n $(n > 2)$, vanishes, then*

$$\bar{\pi}^{ik}\bar{\pi}^{ab}\left(g_{ia}K_{bk} - \frac{1}{n-1}g_{ik}K_{ab}\right) = \text{const}\ .$$

Let us now suppose that Riemannian space V_n is a hypersurface of an Euclidean space E_{n+1}. Then the equations of Gauss and Codazzi have the forms [4]:

$$K_{ihkj} = \Omega_{ik}\Omega_{hj} - \Omega_{ij}\Omega_{hk} \tag{5.5}$$
$$\nabla_k\Omega_{ij} - \nabla_j\Omega_{ik} = 0\ . \tag{5.6}$$

If we choose the tensor π_{ij} such that

$$\pi_{ij} = \sqrt{S}\ \Omega_{ij}\quad ,\quad S = \text{const}\ . \tag{5.7}$$

the condition (2.4) is satisfied, and equations of Gauss have the form (5.4). Therefore, we obtain

Theorem . *π-projective curvature tensor of a hypersurface of an Euclidean space E_{n+1} $(n > 2)$, where tensor π has the form* (5.7), *vanishes. In the other words: a hypersurface of an Euclidean space E_{n+1} $(n > 2)$ admits such π-projective transformation that π-projective curvature tensor vanishes.*

Conversely, if the curvature tensor of the Riemannian space has the form (5.5) and the equation (5.6) is satisfied, it is of class one, i.e. it can be immersed in a flat space of $n+1$ dimensions ([4], p.198).

Thus we have

Theorem . *If a Riemannian space admits π-projective transformations such that π-projective curvature tensor vanishes, and constant $S = S_{ab}\bar{\pi}^{ab}$ is positive, it is of class one and its second fundamental tensor has the form*

$$\Omega_{ij} = \frac{1}{\sqrt{S}}\pi_{ij} .$$

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STRESZCZENIE

Autor określa zmiany koneksji Γ na rozmaitości różniczkowalnej, które zachowują układ π-geodetyk — pojęcie to określił on we wcześniejszej pracy [5] — oraz określa przekształcenie π-projektywne.

Przy założeniu symetryczności koneksji Γ i tensora π spełniającego warunek (2.4) znaleziono pierwszy i drugi π-projektywny tensor krzywizny.

SUMMARY

The author defines the transformations of a connection Γ on a differentiable manifold which preserve the system of π-geodesics as defined in [5] and introduces π-projective transformations.

Under the assumption of symmetry of the connection Γ and for the tensor π satisfying (2.4) the first and the second π-projective curvature tensor has been found.