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On The Univalent Holomorphic Maps of the Unit Polydisc in C^n Which Have the Parametric Representation II – the Necessary Conditions and the Sufficient Conditions

O odwzorowaniach jednokrotnych policylindra jednostkowego w C^n mających przedstawienie parametryczne II - warunki konieczne i warunki dostateczne

In this paper we produce the necessary conditions and the sufficient conditions which guarantee that a univalent holomorphic map of the unit polydisc in C^n have the parametric representation.

Let C^n be the space of n complex variables $z = (z_1, \ldots, z_n)$, $z_j \in C$, $j = 1, \ldots, n$. For $z = (z_1, \ldots, z_n) \in C^n$ we define $\|z\| = \max_{1 \le j \le n} |z_j|$. Let $P^n(r) = \{z \in C^n; \|z\| < r\}$ and $P^n = P^n(1)$.

By I, we shall denote the identity map on C^n.

The class of holomorphic maps of a domain Ω (contained in C^n) into C^n is denoted by $H(\Omega)$.

Let $M(P^n(r))$ be the class of maps $h : P^n(r) \to C^n$ which are holomorphic and satisfy the following conditions: $h(0) = 0$, $Dh(0) = I$ and $\mathrm{re}(h_j(z)/z_j) \ge 0$ when $\|z\| = |z_j| > 0$ $(1 \le j \le n)$, where $h = (h_1, \ldots, h_n)$ (compare [2], [8]).

We shall say that the function f from $[s, \infty)$ (where $s \ge 0$) into C^n is almost absolutely continuous on $[s, \infty)$ if it is absolutely continuous on every bounded closed interval contained in $[s, \infty)$.

In this paper we shall study relations between the classes $S(P^n)$ and $S^0(P^n)$.

Definition 1. We shall say that $f \in S(P^n)$ if and only if $f \in H(P^n)$, $f(0) = 0$, $Df(0) = I$ and f is univalent on P^n.

Definition 2. We say that $f \in S^0(P^n)$ if and only if there exists a function $h = h(z, t)$ from $P^n \times [0, \infty)$ into C^n which satisfies conditions:

(i) for every $t \in [0, \infty)$, $h(\cdot, t) \in M(P^n)$

(ii) for every $z \in P^n$, $h(z, \cdot)$ is a measurable function on P^n such that

$$\lim_{t \to \infty} e^t v_0(z, t) = f(z) \qquad \text{for } z \in P^n$$

where $v_0 = v_0(z,t)$ (for $z \in P^n$, $t \geq 0$) is such a solution of the equation

$$(1) \qquad \frac{\partial v_0}{\partial t} = -h(v_0,t) \qquad \text{for a.e.} \quad t \in [0,\infty) \ , \ v_0(z,0) = z$$

that for every $z \in P^n$, $v_0(z,\cdot)$ is an almost absolutely continuous function on $[0,\infty)$.

In [6] it is shown that for $n \geq 2$ $S^0(P^n)$ is proper subclass of the class $S(P^n)$. In this paper we produce the necessary conditions and sufficient conditions which guarantee that a map f from $S(P^n)$ belongs to the class $S^0(P^n)$.

Definition 3. Let Ω be an open subset of $\mathbf{C}^n$ such that $0 \in \Omega$. The set Ω will be called asymptotically starlike if and only if there exists a map Φ from $\Omega \times [0,\infty)$ into $\mathbf{C}^n$ such that

1^0 $\Phi(0,t) = 0$ for $t \in [0,\infty)$,
2^0 for every $t \in [0,\infty)$, $\Phi(\cdot\,,t)$ is holomorphic on Ω and $D\Phi(0,t) = I$,
3^0 for every $\alpha \in \Omega$, $\Phi(\alpha,\cdot)$ is measurable on $[0,\infty)$,
4^0 for every $\alpha \in \Omega$ and $s \geq 0$, the differential equation

$$w'(t) = -\Phi(w,t) \qquad \text{for a.e.} \quad t \geq s \ , \ w(s) = \alpha$$

possesses exactly one almost absolutely continuous solution on $[0,\infty)$ (which further we shall denote by $w = w(\alpha,s,t)$) and differentiable with respect to α on Ω ,
5^o $\quad w(\alpha,0,t) = e^{-t}\alpha + 0_\alpha(t)$
where $\lim\limits_{t\to\infty} e^t 0_\alpha(t) = 0$ and this convergence is almost uniform on Ω.

Definition 4. Let Ω be an open subset of $\mathbf{C}^n$ such that $0 \in \Omega$. The set Ω will be called smoothly asymptotically starlike if and only if it is asymptotically starlike and the map $w = w(\alpha,s,t)$ from the definition 3 is, for all $\alpha \in \Omega$ and $s \geq 0$, differentiable in the point $t = s$.

Remark 1. Let $\Omega \subset \mathbf{C}^n$ be an open set including 0. If Ω is a starlike set, then it is smooothly asymptotically starlike.

Theorem 1. *If $f \in S^0(P^n)$, then $f(P^n)$ is an asymptotically starlike set.*

Proof. Since $f \in S^0(P^n)$ therefore there exists a function $h : P^n \times [0,\infty) \to \mathbf{C}^n$ satisfying conditions (i)–(ii) from definition 2 and such that $\lim\limits_{t\to\infty} e^t v_0(z,t) = f(z)$ for $z \in P^n$, where for every $z \in P^n$, $v_0(z,\cdot)$ is an almost absolutely continuous on $[0,\infty)$ solution of equation (1).

Now, define a function $v = v(z,s,t)$, for $z \in P^n$, $t \geq s \geq 0$ as that in lemma 2 from [6]. It is easy to see that $v(z,0,t) = v_0(z,t)$ for $z \in P^n$ and $t \in [0,\infty)$. Next, let us introduce a function

$$\Phi(\alpha,t) = Df\big(f^{-1}(\alpha)\big) \circ h\big(f^{-1}(\alpha),t\big) \qquad \text{for } \alpha \in f(P^n) \quad \text{and } t \geq 0 \ .$$

We shall show that such defined function Φ is a looked up function. It is not difficult to see that Φ fulfils conditions 1^o and 2^o from definition 3. Since for every $z \in P^n$ the function $h(z,\cdot)$ is measurable on $[0,\infty)$, therefore for every $\alpha \in f(P^n)$ $\Phi(\alpha,\cdot)$ is also measurable on $[0,\infty)$. Let $\alpha \in f(P^n)$ and $s \geq 0$. The function $w = f\big(v(f^{-1}(\alpha),s,t)\big)$, for $t \geq s$, is almost absolutely continuous on $[s,\infty)$. It is not difficult to show that such defined function $w = w(\alpha, s, t)$ fulfils the differential equation

$$w'(t) = -\Phi(w,t) \qquad \text{for a.e. } \ t \geq s\,,\ w(s) = \alpha\,.$$

Since $\lim_{t\to\infty} e^t v_0(z,t) = f(z)$, for $z \in P^n$ (and this convergence is almost uniform on P^n) and $Df(0) = I$, therefore

$$\lim_{t\to\infty} e^t w(\alpha,0,t) = \alpha \qquad \text{for} \quad \alpha \in f(P^n)\,,$$

and this convergence is also almost uniform on $f(P^n)$.

Hence we showed that the function $\Phi : f(P^n) \times [0,\infty) \to \mathbb{C}^n$ satisfies conditions $1^\circ - 5^\circ$ from definition 3 . Consequently, $f(P^n)$ is an asymptotically starlike set.

Corollary . *Let $\Omega \subset \mathbb{C}$ and $0 \in \Omega$. If Ω is a simply connected domain (the definition of the simply connectedness, see e.g. [7]) then there exists $R > 0$ such that $\frac{1}{R}\Omega$ is an asymptotically starlike set.*

Proof. At first let us consider the case when $\Omega = \mathbb{C}$. Then as the map Φ, appearing in definition 3, we can put $\Phi(\alpha,t) = \alpha$ for $\alpha \in \mathbb{C}$ and $t \in [0,\infty)$. When $\Omega \underset{\neq}{\subset} \mathbb{C}$ is a simply connected domain then by Riemann theorem (see theorem 14.8 from [7] and chapter XII, §11 from [3]) there exists such a function F mapping unit polydisc P^1 into Ω that $F(0) = 0$ and $F'(0) = R$, where R is some positive number. Next, let us introduce map $f = \frac{1}{R}F$. Such defined function f belongs to $S(P^1)$. Since $S(P^1) = S^0(P^1)$ (compare [6]) therefore in virtue of theorem 1, $f(P^1)$ is an asymptotically starlike set. Hence $\frac{1}{R}\Omega$ is also an asymptotically starlike set because $f(P^1) = \frac{1}{R}\Omega$.

Theorem 2. *If $f \in S(P^n)$ and $f(P^n)$ is a smoothly asymptotically starlike set then $f \in S^0(P^n)$.*

Proof. Since $f(P^n)$ is asymptotically starlike, therefore there exists a function Φ from $f(P^n) \times [0,\infty)$ into $\mathbb{C}^n$ satisfying conditions $1^\circ - 5^\circ$ from definition 3. Let $w = w(\alpha,s,t)$ for $\alpha \in f(P^n)$, $t \geq s \geq 0$ be a function defined in 4° of definition 3. Of course, $w(\alpha,s,t) \in f(P^n)$ for any $\alpha \in f(P^n)$ and $t \geq s \geq 0$. Let us introduce a function $v = v(z,s,t)$ for $z \in P^n$ and $t \geq s \geq 0$ in the following way $v(z,s,t) = f^{-1}\big(w(f(z),s,t)\big)$. For any $z \in P^n$ and $s \geq 0$ the function $v(z,s,\cdot)$ is almost absolutely continuous on $[s,\infty)$. Hence for almost every $t \in [s,\infty)$ we have

$$\text{(2)} \qquad \frac{\partial v}{\partial t}(z,s,t) = -h\big(v(z,s,t),t\big)\,, \qquad v(z,s,s) = z$$

where $h(z,t) = (Df^{-1})(f(z)) \circ \Phi(f(z))$ for $z \in P^n$ and $t \geq s$. By definition 3 and from the definition of v, it follows that for any $z \in P^n$ and $s \geq 0$, the function $v(z,s,\cdot)$ is differentiable in point $t = s$. Hence from equality (2) we have

$$\text{(3)} \qquad \frac{\partial v}{\partial t}(z,s,s) = -h(z,s) \qquad \text{for } z \in P^n \quad \text{and } s \geq 0\,.$$

Let $s \geq 0$. Define an auxiliary function

$$\tilde{v}(z,\tau) = v(z,s,s+\tau) \qquad \text{for } z \in P^n \quad \text{and } \tau \geq 0 .$$

For any $z \in P^n$ the function $\tilde{v}(z,\cdot)$ is differentiable in point $\tau = 0$ and $\frac{\partial \tilde{v}}{\partial \tau}(z,0) = \frac{\partial v}{\partial t}(z,s,s)$. On the other hand from the definition of the function h we have that $Dh(0,s) = I$ for $s \geq 0$. By the above facts, lemma 1 from [8] and equality (3) it follows that $h(\cdot,s) \in M(P^n)$ for $s \geq 0$. Also from the definition of the function h it appears that for every $z \in P^n$ the function $h(z,\cdot)$ is measurable on $[0,\infty)$.

Since $\lim_{t\to\infty} e^t w(\alpha,0,t) = \alpha$ for any $\alpha \in f(P^n)$ and $Df^{-1}(0) = I$, therefore

$$\lim_{t\to\infty} e^t v(z,0,t) = \lim_{t\to\infty} e^t f^{-1}\big(w(f(z),0,t)\big) = f(z) .$$

for $z \in P^n$. Hence $f \in S^0(P^n)$.

Remark 2. If f is a starlike map (i.e. f is univalent, $f \in H(P^n)$, $f(0) = 0$, $Df(0) = I$ and $f(P^n)$ is a starlike set) then f has a parametric representation i.e. $f \in S^0(P^n)$.

It follows immediately from remark 1 and theorem 2.

Definition 5. A normalized univalent subordination chain is called a map $f : P^n \times [0,\infty) \to \mathbf{C}^n$ such that

1) for every $t \geq 0$, $f(\cdot,t) \in H(P^n)$, $f(0,t) = 0$, $Df(0,t) = \frac{\partial f}{\partial t}(0,t) = e^t I$,

2) for every $t \geq 0$, $f(\cdot,t)$ is univalent on P^n,

3) for every $z \in P^n$, $f(z,\cdot)$ is almost absolutely continuous on $[0,\infty)$,

4) for (s,t) such that $0 \leq s \leq t < \infty$ there exists a Schwarz function $v = v(z,s,t)$ for $z \in P^n$ such that

$$f(z,s) = f\big(v(z,s,t),t\big) \qquad \text{for } z \in P^n .$$

Definition 6. We shall say that the normalized univalent subordination chain is smooth if for any $z \in P^n$ and $s \geq 0$ the function $v(z,s,\cdot)$ has the continuous derivative in a certain right–hand neighbourhood of the point $t = s$.

Theorem 3. *If $f_0 \in S^0(P^n)$ then there exists a normalized univalent subordination chain f such that f_0 is the first element of this chain (i.e. $f(z,0) = f_0(z)$ for $z \in P^n$).*

Proof. Since $f_0 \in S^0(P^n)$ therefore there exists a map h from $P^n \times [0,\infty)$ into $\mathbf{C}^n$ satisfying conditions (i) – (ii) from definition 2 and such that

$$\lim_{t\to\infty} e^t v_0(z,t) = f_0(z) \qquad \text{for } z \in P^n , \tag{4}$$

where for every $z \in P^n$, $v_0(z,\cdot)$ is the almost absolutely continuous on $[0,\infty)$ solution of equation (1).

Let $f = f(z,s)$, for $z \in P^n$ and $s \geq 0$, be defined as that in lemma 3 from [6]. In virtue of lemma 4 from [6] the map $f(z,\cdot)$ is almost absolutely continuous on $[0,\infty)$ for every $z \in P^n$ and the following equality takes place

$$f(z,s) = f\big(v(z,s,t),t\big) \qquad \text{for } z \in P^n \quad \text{and } t \geq s \geq 0\,,$$

where $v = v(z,s,t)$ for $z \in P^n$ and $t \geq s \geq 0$ is such a solution of differential equation

$$\frac{\partial v}{\partial t} = -h(v,t) \qquad \text{for a.e. } \ t \geq s\,, \quad v(z,s,s) = z$$

that for $z \in P^n$ and $s \geq 0$ the function $v(z,s,\cdot)$ is almost absolutely continuous on $[0,\infty)$.

Consequently, by lemma 3 from [6] f is a normalized univalent subordination chain.

Since $v_0(z,t) = v(z,0,t)$ for $z \in P^n$ and $t \geq 0$, therefore it is visible at once that $f(z,0) = f_0(z)$ for $z \in P^n$.

Remark 3. The normalized univalent subordination chain which is constructed in the proof of the above theorem fulfils the inequality

$$\|f(z,s)\| \leq e^s \,\frac{\|z\|}{(1-\|z\|)^3} \qquad \text{for } z \in P^n \quad \text{and } s \geq 0\,.$$

This fact follows from corollary 2 and lemma 3 from [6].

Theorem 4. *Let f be a smooth normalized univalent subordination chain such that there exist $\delta \in (0,1)$, $t_0 > 0$ and $L > 0$ such that $\|f(z,s)\| \leq L \cdot e^s$ for $z \in P^n(\delta)$ and $s > t_0$. Then the first element of this chain belongs to $S^0(P^n)$.*

Proof. By definition of the smooth normalized univalent subordination chain it follows that there exists a function $v = v(z,s,t)$ for $z \in P^n$, $t \geq s \geq 0$ such that

$$\text{(5)} \qquad f(z,s) = f\big(v(z,s,t),t\big) \qquad \text{for } z \in P^n\,, \quad \text{and } t \geq s \geq 0$$

and that for any $z \in P^n$ and $s \geq 0$ the function $v(z,s,\cdot)$ has the continuous derivative in a certain right-hand neighbourhood of the point $t = s$.

Next let

$$h(z,s) = -\frac{\partial v}{\partial t}(z,s,s) \qquad \text{for } z \in P^n\,, \ s \geq 0\,.$$

Since $Df(0,t) = e^t I$, therefore by equality (5) we get that $Dv(0,s,t) = e^{s-t} I$, i.e. $Dv(0,s,s) = I$ for $t \geq s \geq 0$. Behaving analogously as in the proof of theorem 2 we conclude that $h(\cdot\,,s) \in M(P^n)$ for any $s \geq 0$.

Differentiating equality (5) with respect to the parameter t, we obtain for $z \in P^n$, $s \geq 0$ and $t = s$

$$\text{(6)} \qquad Df(z,s) \circ h(z,s) = \frac{\partial f}{\partial t}(z,s)\,.$$

Let $z \in P^n$, $s \geq 0$ and $\tilde{v}(z,s,\cdot)$ be an almost absolutely continuous solution of the equation

$$(7) \qquad \frac{\partial \tilde{v}}{\partial t} = -h(\tilde{v},t) \quad \text{for a.e. } t \geq s , \quad \tilde{v}(z,s,s) = z .$$

Consider an auxiliary function

$$g(t) = f\big(\tilde{v}(z,s,t),t\big) \qquad \text{for } t \geq s .$$

Such defined the function g is almost absolutely continuous on $[s,\infty)$ and $g'(t) = 0$ for a.e. $t \geq s$ on the ground of equality (6). Hence g is constans on $[s,\infty)$, so

$$(8) \qquad f(z,s) = f\big(v(z,s,t),t\big) \qquad \text{for } z \in P^n \quad \text{and } t \geq s .$$

Since $\|f(z,s)\| \leq L \cdot e^s$ for $z \in P^n(\delta)$ and $s > t_0$, where δ is a certain number from $(0,1)$ therefore proceeding analogously as in the proof of theorem 2 from [5] we get that there exists $\delta_0 > 0$ such that for $a \in P^n(\delta_0)$ and $s > t_0$

$$(9) \qquad \left\|\frac{1}{2!}\, D^2 f(a,s)(z,z)\right\| \leq \frac{L\, e^s}{\delta_0^2}\, \|z\|^2 \qquad \text{for } z \in \mathbb{C}^n .$$

By Taylor formula (see [1]) we have

$$(10) \qquad f(z,s) = e^s z + \int_0^1 (1-\tau) D^2 f(\tau z,s)(z,z)\, d\tau \qquad \text{for } z \in P^n , \quad s \geq 0 .$$

Taking (8), (9) and (10) into account, we obtain that there exists $t_1 \geq t_0$ such that for $s > t_1$ and $z \in P^n$

$$f\big(\tilde{v}(z,0,s),s\big) = e^s v(z,0,s) + r(\tilde{v},s) ,$$

where

$$\|r(\tilde{v},s)\| \leq L \cdot e^s \|\tilde{v}(z,0,s)\| \qquad \text{for } s > t_1 .$$

In virtue of this, corollary 2 and lemma 3 from [6] and by equality (8) we get that

$$f(z,0) = \lim_{t \to \infty} e^t \tilde{v}(z,0,t) \qquad \text{for } z \in P^n .$$

Hence $f(\cdot,0) \in S^0(P^n)$ and this ends the proof.

Remark 4. Let f be a close–to–starlike function relative to the starlike function g. Then, in accordance with theorem 1 from [4], the map

$$F(z,t) = f(z) + \big(e^t - 1\big) g(z) \qquad \text{where } z \in P^n , \quad t \geq 0$$

is univalent subordination chain satisfying the assumptions of theorem 4. Hence the function f being the first element of this chain has the parametric representation.

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STRESZCZENIE

W pracy tej zostały podane pewne warunki konieczne i pewne warunki dostateczne na to, aby jednokrotne holomorficzne odwzorowanie policylindra jednostkowego w $\mathbb{C}^n$ posiadało przedstawienie parametryczne.

SUMMARY

In this paper some necessary and some sufficient conditions for univalent holomorphic mappings of the unit polydisk in $\mathbb{C}^n$ into $\mathbb{C}^n$ to have a parametric representation are given.