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A Sewing Theorem for Complementary Jordan Domains

Twierdzenie o zszywaniu dla komplementarnych obszarów Jordana

Introduction. Let Γ be a Jordan curve in the finite plane and let $D, D^* \ni \infty$ be complementary domains of Γ. Due to the Riemann and Taylor–Osgood–Caratheodory theorems there exist homeomorphisms $h : \overline{\Delta} \to \overline{D}$ and $h^* : \overline{\Delta^*} \to \overline{D^*}$ conformal in $\Delta = \{z : |z| < 1\}$ and $\Delta^* = \{z : |z| > 1\} \cup \{\infty\}$, respectively. The composition $\gamma = (h^*)^{-1} \circ h$ is an automorphism of $\partial\Delta := T$, i.e. a sense-preserving homeomorphism of T onto itself. In this paper we prove a theorem which gives sufficient conditions for an automorphism γ of T to admit the above given representation $(h^*)^{-1} \circ h$ for some Jordan curve Γ. This theorem is essentialy an analogue of the theorem proved by Pfluger [6] and Lehto–Virtanen [4], [5] for the upper and lower half-plane instead of Δ and Δ^*, when γ is a quasisymmetric function of the real axis R. The proof of our theorem is based on a method similar to that used by Lehto and Virtanen but instead of the Beurling–Ahlfors theorem [1] its counterpart for T due to Krzyż [2], [3] has been applied.

We denote by A_T the class of all automorphisms of T which keep the point 1 fixed and have a conformal extension on some annulus $\{z : r < |z| < R\}$, $r < 1 < R$.

Definition . An automorphism $\gamma : T \to T$ is said to be M-quasisymmetric, $M \geq 1$, iff the inequality

$$M^{-1} \leq |\gamma(l_1)|/|\gamma(l_2)| \leq M$$

holds for each pair of adjacent closed arcs $l_1, l_2 \subset T$ such that $0 < |l_1| = |l_2| \leq \pi$, where $|\cdot|$ denotes the Lebesgue measure on T.

The family of all M-quasisymmetric automorphisms of T will be denoted by Q_T^M.

Lemma . *If $\gamma \in A_T \cap Q_T^M$, $M \geq 1$, then there exist K-quasiconformal automorphisms h, h^* of the closed plane $\bar{C}$, where the constant K depends only on M, and an analytic Jordan curve Γ such that*

(i) *h and h^* are conformal in Δ and Δ^* respectively;*
(ii) *$h(\Delta)$ and $h^*(\Delta) \ni \infty$ are complementary domains of Γ;*
(iii) *$h(z) = h^* \circ \gamma(z)$ for $z \in T$;*

(iv) $h(z) = h^*(z) = z$ *for* $z = 0, 1, \infty$.

Proof. Let $\gamma : P \to C$ be conformal in some annulus $P = \{z : r < |z| < R\}$, $r < 1 < R$, where $\gamma|T \in A_T \cap Q_T^M$. By the J.Krzyż theorem [2], [3] there exists a K-quasiconformal extension φ of γ on the whole closed plane $\hat{C}$ which keeps the points $0, 1, \infty$ fixed and the constant K depends only on M. Now we will construct by induction a sequence (φ_n) mappings as follows. Let Δ_t be the disc with the centre at 0 and radius $t > 0$ whose boundary is denoted by T_t. Let $P_1 = \{z : r < |z| < 1\}$. There exist R_1, $0 < R_1 < 1$, and a homeomorphism $\psi_1 : \Delta^* \cup \gamma(\overline{P}_1) \to \hat{C} \setminus \Delta_{R_1}$ conformal in the domain $\overline{\Delta^*} \cup \gamma(P_1)$ such that $\psi_1(z) = z$ for $z = 1, \infty$. We define

$$\varphi_1(z) = \begin{cases} \psi_1 \circ \varphi(z)\,, & z \in \Delta^* \\ \psi_1 \circ \gamma(z)\,, & z \in \overline{P}_1 \\ R_1^2\big(\overline{\varphi}_1(r_1^2\,\overline{z}^{-1})\big)^{-1}\,, & z \in \Delta_{r_1} \end{cases}$$

where $r_1 = r$. Let for every $n \in \mathbf{N}$, $\mathbf{N}$ is the set of natural numbers, $P_{n+1} = P_n \cup T_{r_n} \cup \cup P_n^*$, where P_n^* is an annulus symmetric to P_n with respect to the circle T_{r_n}. There exist R_{n+1}, $0 < R_{n+1} < 1$ and a homeomorphism $\psi_{n+1} : \varphi_n(\Delta^* \cup \overline{P}_{n+1}) \to \hat{C} \setminus \Delta_{R_{n+1}}$ conformal in the domain $\varphi_n(\overline{\Delta^*} \cup P_{n+1})$ such that $\psi_{n+1}(z) = z$ for $z = 1, \infty$.

We define

$$\varphi_{n+1}(z) = \begin{cases} \psi_{n+1} \circ \varphi_n(z)\,, & z \in \hat{C} \setminus \Delta_{r_{n+1}} \\ R_{n+1}^2\big(\overline{\varphi}_{n+1}(r_{n+1}^2\,\overline{z}^{-1})\big)^{-1}\,, & z \in \Delta_{r_{n+1}} \end{cases}$$

where $r_{n+1} = r_n^2$. By the reflection principle for quasiconformal mappings [5] and by induction we obtain that for every $n \in \mathbf{N}$, φ_n is a K-quasiconformal mapping of $\hat{C}$ onto itself which keeps the points $0, 1, \infty$ fixed. Hence the family $\{\varphi_n : n \in \mathbf{N}\}$ is normal [5] and there exists a subsequence (φ_{n_k}) of the sequence (φ_n) which almost uniformly converges to some K-quasiconformal mapping φ_0 of $\hat{C}$ onto itself. Of course $\varphi_0(z) = z$ for $z = 0, 1, \infty$.

From the definition of φ_n it follows that $\varphi_n(z) = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_1 \circ \varphi(z)$ for $n \in \mathbf{N}$ and $z \in \Delta^*$. Hence the sequence $\varphi_{n_k} \circ \varphi^{-1}$, $k \in \mathbf{N}$ converges to the conformal mapping $\varphi_0 \circ \varphi^{-1}$ in the domain Δ^*. By the reflection principle for conformal mappings and by induction we obtain that φ_n is conformal in the annulus P_{n+1} for $n \in \mathbf{N}$. Since $P_n \subset P_{n+1}$ for $n \in \mathbf{N}$, $\bigcup_{n=1}^{\infty} P_n = \Delta \setminus \{0\}$, the sequence (φ_{n_k}) is uniformly convergent to the mapping φ_0 in Δ and $\varphi_0(0) = 0$, φ_0 being conformal in Δ. Putting $h = \varphi_0$, $h^* = \varphi_0 \circ \varphi^{-1}$ and $\Gamma = \varphi_0(T)$ we see that h and h^* are K-quasiconformal mappings of $\hat{C}$ onto itself, conformal in Δ and Δ^*, respectively. Moreover, $h(\Delta) = \varphi_0(\Delta)$ and $h^*(\Delta^*) = \varphi_0(\Delta^*) \ni \infty$ are complementary domains of $\Gamma = \varphi_0(T)$, $h(z) = h^*(z) = z$ for $z = 0, 1, \infty$ and for every point $z \in T$ we have $h^* \circ \gamma(z) = \varphi_0 \circ \varphi^{-1} \circ \varphi(z) = \varphi_0(z) = h(z)$. The function $\lambda : P \to C$ defined by the formula

$$\lambda(z) = \begin{cases} h(z)\,, & z \in P \cap \overline{\Delta} \\ h^* \circ \gamma(z)\,, & z \in P \cap \Delta^* \end{cases}$$

is analytic in $P \setminus T$ and continuous in P. Consequently λ is analytic in the whole annulus P. Then from the equality $\Gamma = \varphi_0(T) = \lambda(T)$ it follows that Γ is an analytic Jordan curve and this ends the proof.

As a consequence of the above lemma we obtain

Theorem . *If $\gamma \in Q_T^M$, $M \geq 1$ then there exist K-quasiconformal mappings h, h^* of the closed plane $\bar{C}$ onto itself, where the constant K depends only on M and a Jordan curve Γ such that*

(i) *h and h^* are conformal in Δ and Δ^*, respectively ;*

(ii) *$h(\Delta)$ and $h^*(\Delta^*) \ni \infty$ are complementary domains of Γ ;*

(iii) *$h(z) = h^* \circ \gamma(z)$ for $z \in T$;*

(iv) *$h(z) = h^*(z)$ for $z = 0, \infty$.*

Proof. Without loss of generality we may assume that $\gamma(1) = 1$. This can be always achieved after a suitable rotation. There exists a homeomorphism σ of R onto itself such that $\sigma(0) = 0$, $\sigma(t + 2\pi) = 2\pi + \sigma(t)$ and $\gamma(e^{it}) = e^{i\sigma(t)}$ for every $t \in R$. Let $n \in N$ be arbitrary and put $P_n(z) = cn(n^4z^4 + 1)^{-1}$ for $z \in D_n$, where $D_n = \{z : |\mathrm{Im}\, z| < 1/2n\}$ and $c^{-1} = \int\limits_{-\infty}^{\infty} (x^4 + 1)^{-1}\, dx$. We define the function $\sigma_n : D_n \to C$ by the following formula $\sigma_n(z) = P_n * \sigma(z) - P_n * \sigma(0)$, where $P_n * \sigma(z) = \int\limits_{-\infty}^{\infty} P_n(z - t)\, \sigma(t)\, dt$.

Since $P_n'(z) = -4cn^5z^3(n^4z^4 + 1)^{-2}$ for $z \in D_n$, σ_n is an analytic function in the strip D_n and

$$\sigma_n'(z) = P_n' * \sigma(z) . \tag{1}$$

It is easy to verify, using for example the Cauchy integral theorem, that for every $z \in D_n$, $\int\limits_{-\infty}^{\infty} P_n(z - t)\, dt = 1$. Hence for every $z \in D_n$ we have

$$\begin{aligned} \sigma_n(z + 2\pi) &= P_n * \sigma(z + 2\pi) - P_n * \sigma(0) = \\ &= P_n * \sigma(z) - P_n * \sigma(0) + 2\pi \int_{-\infty}^{\infty} P_n(z - t)\, dt = \sigma_n(z) + 2\pi . \end{aligned} \tag{2}$$

From (1) we obtain for every $x \in R$

$$\sigma_n'(x) = 4cn^5 \int_{-\infty}^{\infty} (t - x)^3 \left(n^4(x - t)^4 + 1\right)^{-2} \sigma(t)\, dt > 0$$

as the homeomorphism σ is increasing . By this and (2) there exists ϵ_n , $0 < \epsilon_n \leq 1/2n$ such that $\mathrm{Re}\, \sigma_n'(z) > 0$ for $z \in \tilde{D}_n = \{z : |\mathrm{Im}\, z| < \epsilon_n\}$. Thus putting $R_n = \{z : |\log|z|| < \epsilon_n\}$ we see that the mapping $\gamma_n : R_n \to C$, $\gamma_n(z) = \exp(i\sigma_n(-i \log z))$ for $z \in R_n$, is conformal in the annulus R_n, so $\gamma_n|T \in A_T$. Since $\gamma \in Q_T^M$, the following inequality holds for any $x \in R$ and $y \in (0, \pi]$: $M^{-1}(\sigma(x) - \sigma(x - y)) \leq \sigma(x + y) - \sigma(x) \leq M(\sigma(x) - \sigma(x - y))$.

Consequently

$$\sigma_n(x+y) - \sigma_n(x) = P_n * \sigma(x + y) - P_n * \sigma(x) = \int_{-\infty}^{\infty} P_n(x - t)\left(\sigma(t+y) - \sigma(t)\right) dt \leq$$

$$\leq M \int_{-\infty}^{\infty} P_n(x-t)\left(\sigma(t) - \sigma(t-y)\right) = M\left(P_n * \sigma(x) - P_n * (x-y)\right) = M\left(\sigma_n(x) - \sigma_n(x-y)\right)$$

and similarly

$$\sigma_n(x+y) - \sigma_n(x) \geq M^{-1}\big(\sigma_n(x) - \sigma_n(x-y)\big) .$$

Finally for every $n \in \mathbf{N}$ the homeomorphism $\gamma_n|T \in A_T \cap Q_T^M$ and

$$\sigma_n(t) \longrightarrow \sigma(t) \quad \text{as} \quad n \longrightarrow \infty \quad \text{for every} \quad t \in R . \tag{3}$$

By the lemma there exist sequences $(h_n), (h_n^*)$ of K-quasiconformal mappings of $\hat{C}$ onto itself which keep the points $0, 1, \infty$ fixed. Thus the families of mappings $\{h_n : n \in \mathbf{N}\}$, $\{h_n^* : n \in \mathbf{N}\}$ are normal [5] and there exist subsequences $(h_{n_k}), (h_{n_k}^*)$, which are almost uniformly convergent to K-quasiconformal mappings h and h^* of $\hat{C}$ onto itself, respectively. Moreover, $h(z) = h^*(z) = z$ for $z = 0, 1, \infty$. By (3) it follows that $\gamma_n(z) \to \gamma(z)$ as $n \to \infty$. Since $h_n(z) = h_n^*(z) \circ \gamma_n(z)$ for $n \in \mathbf{N}$, we have for every $z \in T$: $h(z) = \lim_{k\to\infty} h_{n_k}(z) = \lim_{k\to\infty} h_{n_k}^*\big(\gamma_{n_k}(z)\big) = h^* \circ \gamma(z)$. Hence $h(T) = h^*(T)$ and $h(\Delta), h^*(\Delta^*) \ni \infty$ are complementary domains of a Jordan curve $\Gamma = h(T)$. Moreover, h and h^* are conformal mappings in Δ and Δ^*, respectively, as limits of almost uniformly convergent sequences of conformal mappings in Δ and Δ^*. This ends the proof.

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STRESZCZENIE

Wykazano, że każdy k-quasisymetryczny automorfizm γ okręgu jednostkowego T jest konforemną reprezentacją pewnego K-quasiokręgu Γ, analitycznego jeśli γ jest automorfizmem analitycznym, gdzie stała K zależy tylko od stałej k.

SUMMARY

The author proves that any k-quasisymmetric automorphism γ of the unit circle T is a conformal representation of a K-quasicircle Γ which is analytic as soon as γ is analytic ; the constant K depends on k only.