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# On Some Sufficient Conditions for $\alpha$-convexity of Order $\beta$

O pewnych warunkach dostatecznych $\alpha$-wypukłości rzędu $\beta$

Let $A$ denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are regular in $E = \{z : |z| < 1\}$ and let $S$ be the subclass of functions from $A$ which are univalent in $E$.

For the function $f \in A$ for which $f(z)f'(z) \neq 0$ , $0 < |z| < 1$ , and

$$\mathrm{Re}\left\{\alpha\left(1 + z\frac{f''(z)}{f'(z)}\right) + (1-\alpha)z\frac{f'(z)}{f(z)}\right\} > \beta\,, \qquad z \in E\,, \tag{1}$$

for some real numbers $\alpha$ and $\beta$ , $0 \leq \beta < 1$ , we say that it is $\alpha$-convex of order $\beta$ in $E$. We denote the class of such functions by $M(\alpha,\beta)$. For $\beta = 0$ we have the class of $\alpha$-convex functions which was introduced by P. Mocanu [3]. It is evident that

$$M(0,\beta) \equiv S^*(\beta)\,, \quad M(0,0) \equiv S^*\,, \quad M(1,\beta) \equiv K(\beta)\,, \quad M(1,0) \equiv K\,,$$

where $S^*(\beta), S^*, K(\beta), K$ denote the classes of starlike functions of order $\beta$, of starlike functions, of convex functions of order $\beta$ and of convex functions, respectively. In that sense the sets $M(\alpha,\beta)$ give a "continuous" passage from convex functions to starlike functions. Moreover, it is true that if $f \in M(\alpha,0)$ , $\alpha \geq 1$ , then $f \in K$ and if $f \in M(\alpha, 0)$ , $\alpha < 1$ , then $f \in S^*$ (see [2],[1]).

Let $f \in S$ and let $\phi(z) = b_1 z + b_2 z^2 + \cdots$ be regular in $E$. Then the function $\phi$ is called the subordinate to the function $f$ if $\phi(E) \subset f(E)$. It is well-known that in this case there exists a regular function $\omega(z)$ , $z \in E$ , for which $|\omega(z)| \leq |z| < 1$ and $\phi(z) = f(\omega(z))$ , $z \in E$. For this relation the following symbol $\phi \prec f$ is used.

In this paper we give some sufficient conditions for a function $f \in A$ to be in the class $M(\alpha,\beta)$. This is essentially the addition to the papers [4] and [5]. First we cite the following result of Robertson [6].

**Theorem A.** *Let $f \in S$. For each $0 \leq t \leq 1$ let $F(z,t)$ be regular in $E$, let $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let $\rho$ be a positive real number for which*

$$F(z) = \lim_{t \to +0} \frac{F(z,t) - F(z,0)}{z\, t^{\rho}}$$

*exists. Let $F(z,t)$ be subordinate to $f(z)$ in $E$ for $0 \le t \le 1$. Then*

$$\operatorname{Re} \frac{F(z)}{f'(z)} \le 0\,, \qquad z \in E\,.$$

*If in addition $F(z)$ is also regular in $E$ and $\operatorname{Re} F(0) = 0$, then*

$$\operatorname{Re} \frac{F(z)}{f'(z)} < 0\,, \qquad z \in E\,. \tag{2}$$

**Theorem 1.** *Let $f \in A$ and let $f(z)f'(z) \ne 0$ for $0 < |z| < 1$. If*

$$g(z) = \int_0^z \frac{f(s)}{s}\left[\frac{s f'(s)}{f(s)}\right]^{\alpha} ds \in S\,, \qquad \alpha \text{ is real}, \tag{3}$$

*(where with the function $\left[\frac{z f'(z)}{f(z)}\right]^{\alpha}$ we select the principal values) and if*

(a) $G_1(z,t) = g(ze^{it}) + g(ze^{-it}) - g(ze^{-\beta t^2}) \prec g(z)\,, \qquad z \in E\,;$

*or*

(b) $G_2(z,t) = \dfrac{1}{1-\beta}\left[\dfrac{1}{2}\left(g(ze^{it}) + g(ze^{-it})\right) - \beta g\left(\left(1-\dfrac{t^2}{2}\right)z\right)\right] \prec g(z)\,, \qquad z \in E$

*for fixed $\alpha$ and $\beta$, $0 \le \beta < 1$ and for each $0 \le t \le 1$, then $f \in M(\alpha,\beta)$.*

**Proof.** It is easy to show that the following implications

$$f \in M(\alpha,\beta) \iff F(z) = f(z)\left[\frac{z f'(z)}{f(z)}\right]^{\alpha} \in S^*(\beta) \iff g(z) = \int_0^z \frac{F(s)}{s}\, ds \in K(\beta)$$

are true. Because of that it is sufficient to show that if $g$ satisfies (1) and (a) or (b), then $g \in K(\beta)$.

First let (1) and (a) be assumed to be true. It is evident that $G_1(z,0) \equiv g(z)$ and $G_1(0,t) \equiv 0$. If in Theorem A we choose $p = 2$ and for the function $F(z,t)$ we take the function $G_1(z,t)$, then after the denotations $G_1^{(1)}(z,t) = g(ze^{it}) + g(ze^{-it})$ and $G_1^{(2)}(z,t) = g(ze^{-\beta t^2})$ we have

$$\begin{aligned} G_1(z) &= \lim_{t \to +0} \frac{G_1(z,t) - G_1(z,0)}{zt^2} = \\ &= \lim_{t \to +0} \left\{ \frac{G_1^{(1)}(z,t) - G_1^{(1)}(z,0)}{zt^2} - \frac{G_1^{(2)}(z,t) - G_1^{(2)}(z,0)}{zt^2} \right\} = \\ &= \lim_{t \to +0} \frac{\partial^2 G_1^{(1)}(z,t)/\partial t^2}{2z} - \lim_{t \to +0} \frac{\partial G_1^{(2)}(z,t)/\partial t}{2zt} = \\ &= -\left[g'(z) + zg''(z) - \beta g'(z)\right]. \end{aligned}$$

Since $G_1(z)$ is regular in $E$ and $\operatorname{Re} G_1(0) = -(1-\beta) \ne 0$, then according to (2) in Theorem A we obtain

$$\operatorname{Re} \frac{G_1(z)}{g'(z)} < 0\,, \qquad z \in E\,,$$

what is equivalent to

$$\operatorname{Re}\left\{1 + z\frac{g''(z)}{g'(z)}\right\} > \beta\,, \qquad z \in E\,,$$

i.e. $g \in K(\beta)$ .

The proof for the case (b) is similar and it may be found in [4] (we note that in the cited paper there exists some typing mistakes, but it is not difficult to remove them).

**Corollary 1.** *For $\alpha = 1$ from* Theorem 1 *we have that the following conditions:*

$(3_1)$ $f \in S$ ;

$(a_1)$ $f(ze^{it}) + f(ze^{-it}) - f(ze^{-\beta t^2}) \prec f(z)\,, \qquad z \in E$ ;

*or*

$(b_1)$ $\dfrac{1}{1-\beta}\left[\dfrac{f(ze^{it}) + f(ze^{-it})}{2} - \beta f((1 - \tfrac{t^2}{2}))\right] \prec f(z)\,, \qquad z \in E\,,$

*are sufficient for convexity of order $\beta$. Hence, especially, for $\beta = 0$ we have that $f \in S$ is a convex function in $E$ if for each $0 \le t \le 1$ :*

$(a_2)$ $f(ze^{it}) + f(ze^{-it}) - f(z) \prec f(z)\,, \qquad z \in E$ ;

*or*

$(b_2)$ $\dfrac{f(ze^{it}) + f(ze^{-it})}{2} \prec f(z)\,, \qquad z \in E$ (Robertson [6]).

From Theorem 1 we can get the corresponding sufficient conditions for starlikeness choosing the appropriate $\alpha$ and $\beta$.

**Theorem 2.** *Let $0 \le \beta < 1$ and $\beta \le \alpha \le 1$. Let $f \in A$ and let $f(z)f'(z) \ne 0$ for $0 < |z| < 1$. If*

$$h(z) = \int_0^z \frac{f(s)f'(s)}{s}\,ds \in S \tag{4}$$

*and if*

$$H(z,t) = f(ze^{-t})f'(ze^{-t}) - f(ze^{-\alpha t})f'(ze^{-(1-\alpha)t}) + h(ze^{-(\alpha-\beta)t}) \prec h(z)\,, \qquad z \in E\,, \tag{5}$$

*for fixed $\alpha$ and $\beta$ and for each $0 \le t \le 1$ , then $f \in M(\alpha, \beta)$ .*

**Proof.** It is evident that $H(z,0) \equiv h(z)$ and $H(0,t) \equiv 0$. By applying Theorem A (choosing $p = 1$) we have that

$$\begin{aligned} H(z) &= \lim_{t\to+0} \frac{H(z,t) - H(z,0)}{zt} = \lim_{t\to+0} \frac{\partial H(z,t)/\partial t}{z} = \\ &= -\left[\alpha f(z)f''(z) + (1-\alpha)\left(f'(z)\right)^2 + (\alpha - \beta)\frac{f(z)f'(z)}{z}\right]\,, \end{aligned} \tag{6}$$

while

$$h'(z) = \frac{f(z)f'(z)}{z}\,.$$

From (6) we have that $H(z)$ is regular in $E$ and $\mathrm{Re}\, H(0) = -(1-\beta) \neq 0$. Then in accordance with Theorem A we have that

$$\mathrm{Re}\, \frac{H(z)}{h'(z)} < 0\,, \qquad z \in E\,,$$

what is equivalent to (1) , i.e. $f \in M(\alpha,\beta)$.

**Corollary 2.** *For $\alpha = 1$ and $\beta = 0$ we have that the condition (5) has the form*

$$f(ze^{-t})[f'(ze^{-t}) - f'(z)] + h(ze^{-t}) \prec h(z)\,, \qquad z \in E\,,$$

*which together with (4) is sufficient for $f \in A$ to be convex in $E$.*

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## STRESZCZENIE

W pracy tej podano pewne warunki dostateczne na to, by funkcja $f(z) = z + a_2 z^2 + \cdots$ regularna w kole $|z| < 1$, była $\alpha$–wypukłą rzędu $\beta$ $(\alpha \in \mathbb{R}\,,\ 0 \leq \beta < 1)$. W szczególności otrzymano warunki gwiaździstości i wypukłości. W dowodach posłużono się metodą podporządkowania.

## SUMMARY


Sufficient conditions for $f(z) = z + a_2 z^2 + \cdots$ holomorphic in $|z| < 1$ to be $\alpha$ – convex of order $\beta$ $(\alpha \in \mathbb{R},\ 0 \leq \beta < 1)$ are given. In particular sufficient conditions of starlikeness and convexity are given. Subordination principle is used in proofs.