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On the Inner Structure of Some Class of Univalent Functions

O wewnętrznej strukturze funkcji spełniających pewien warunek jednolistności

1. Let H denote the class of functions holomorphic in E, where $E_r = \{z : |z| < r\}$, $E_1 = E$, $\bar{E}$ – the closure of E and $S_0 \subset H$ be the subclass of univalent functions in E. Denote by $\Omega_0 \subset H$ the class of functions ω such that $|\omega(z)| \le 1$, $\omega \neq 1$ for $z \in E$.

L.V.Ahlfors [1] and J.Becker [2] give a following sufficient condition for univalence:

Theorem A–B. *If $f \in H$, $f'(0) \neq 0$ and there exists a constant $c \in \bar{E} \setminus \{1\}$ such that*

$$\left| (1 - |z^2|) \frac{z f''(z)}{f'(z)} - c|z|^2 \right| \le 1 , \tag{1.1}$$

then $f \in S_0$.

In the case $c = 0$ this theorem was given earlier by Duren , Shapiro and Shields [3].

The following generalization of the theorem A–B was obtained by Z.Lewandowski [4]:

Theorem L. *Let $f \in H$, $f'(0) \neq 0$. If there exists a function $\omega \in \Omega_0$ such that*

$$\left| \omega(z)|z|^2 - (1 - |z|^2) \left(\frac{z\, \omega'(z)}{1 - \omega(z)} + \frac{z f''(z)}{f'(z)} \right) \right| \le 1 \tag{1.2}$$

for $z \in E$, then $f \in S_0$.

The purpose of this note is to characterize the structure of functions which satisfy the assumptions of theorem L.

The purpose of this note is to characterize the structure of functions which satisfy the assumptions of theorem L.

Denote the class of these functions by $\bar{S}_0$. In the second part of this note we will show that many known subclasses of the class S_0 are contained in the class $\bar{S}_0$.

In the third part we are going to give a K–quasi conformal (K–$q.c.$) extension for some subclass of the class $\bar{S}_0$. At the end of this note we pay some attention to a structural formula for the class $\bar{S}_0$, which seems to be more convenient, for our purpose than inequality (1.2), as it follows from the second part of this paper, in studying extremal problems for the class $\bar{S}_0$.

2. Let $f \in H$ satisfy the equation

$$\frac{z\,\omega'(z)}{1-\omega(z)} + \frac{z\,f''(z)}{f'(z)} = \varphi(z)\,, \qquad z \in E\,, \tag{2.1}$$

for an arbitrary fixed function $\omega \in \Omega_0$ and the function φ satisfying the conditions of Schwarz's Lemma. The class of these functions we denote by $\hat{S}_0$.

From (2.1) we obtain at once

$$f'(z) = \frac{1-\omega(z)}{1-\omega(0)} \exp\left(\int_0^z \frac{\varphi(t)}{t}\,dt\right), \qquad z \in E\,. \tag{2.2}$$

Let

$$g(z) = z\ \exp\left(\int_0^z \frac{\varphi(t)}{t}\,dt\right) \qquad z \in E\,.$$

It is easy to see that

$$\left|\frac{z\,g'(z)}{g(z)} - 1\right| < 1\,, \qquad z \in E$$

then g belongs to the known subclass of the class $S_0^* \subset S_0$ of the functions starlike with respect to the origin.

The relation (2.2) can be rewritten in the form

$$\frac{z\,f'(z)}{g(z)} = \frac{1-\omega(z)}{1-\omega(0)}\,. \tag{2.3}$$

It follows at once from relation (2.1) that inequality (1.2) holds, then $\hat{S}_0 \subset \bar{S}_0$.

In particular, the class $\hat{S}_0$ contains known subclasses of the class of univalent functions.

1^0. Let $\varphi(z) = -\omega(z)$ and $f(0) = 0$. Then $f = g$, where g was given earlier in this note.

2^0. If we put $\omega(0) = 0$ into (2.3), then we obtain some subclass of the class of the functions close–to–convex contained in $\hat{S}_0 \subset \bar{S}_0$.

3^0. Putting into (2.3) $\omega(z) = c \in \bar{E} \setminus \{1\}$, where c is a constant, we obtain $zf' = g$, hence f is a convex function.

3. Let C denote the complex plane, $\bar{C} = C \cup \{\infty\}$, and let $S \subset S_0$ denote the class of functions f such that : $f(0) = 0$, $f'(0) = 1$. Denote by S_K the class of mappings $F : C \to \bar{C}$ K-$q.c.$, such that $F|_E = f \in S$. The symbol $F|_E$ denotes the restriction of the function F to the set E.

It is well known (see [5] p.149) that the class S_K is a compact family with respect to the topology of uniform convergence on compact sets.

Let $\tilde{S}_0(K) \subset \tilde{S}_0 \subset S$ denote the class of functions satisfying the condition:

$$\left|\omega(z)|z|^2 - (1-|z|^2)\left(\frac{z\,\omega'(z)}{1-\omega(z)} + \frac{z\,f''(z)}{f'(z)}\right)\right| \le k < 1\,, \tag{3.1}$$

where $\omega \in \Omega_0$ and $|\omega(z)| \le k$, $z \in E$ and $K = \dfrac{1+k}{1-k}$.

Ahlfors (see [5] p.169) gave for the subclass of the class $\tilde{S}_0(K)$ generated by inequality (3.1) with the function ω equal to a constant $c \in E \setminus \{1\}$, K-$q.c.$ extension $F \in S_k$ such that $F(\infty) = \infty$.

Now we will give an analogous result for the class $\tilde{S}_0(K)$ using the idea of Ahlfors, but with some modification.

To this purpose we give a lemma, which we use in the proof of a suitable theorem.

Lemma . *If $f \in \tilde{S}_0(K)$, then*

$$f_r(z) = \frac{1}{r}f(rz) \in \tilde{S}_0(K), \qquad r \in (0,1)\,.$$

Proof. Let $\omega_r(z) = \omega(rz)$ and

$$A_r(z) = \frac{z\,f_r''(z)}{f_r'(z)} + \frac{z\,\omega_r'(z)}{1-\omega_r(z)}\,.$$

By (3.1) we get

$$\left|A_r(z) - \omega_r(z)\frac{r^2|z|^2}{1-r^2|z|^2}\right| \le \frac{k}{1-r^2|z|^2}$$

Hence

$$\left|A_r(z) - \omega_r(z)\frac{|z|^2}{1-|z|^2} - \omega_r(z)|z|^2\frac{r^2-1}{(1-r^2|z|^2)(1-|z|^2)}\right| \le \frac{k}{1-r^2|z|^2}\,.$$

Thus

$$\left|A_r(z) - \omega_r(z)\frac{|z|^2}{1-|z|^2}\right| \le |\omega_r(z)||z|^2\frac{1-r^2}{(1-r^2|z|^2)(1-|z|^2)} + \frac{k}{1-r^2|z|^2} \le$$

$$\le \frac{k}{1-|z|^2} \qquad \text{because } |\omega_r(z)| \le k\,.$$

Then the function $f_r(z)$ satisfies the inequality (3.1), where instead of $\omega(z)$ we put $\omega_r(z) = \omega(rz)$.

Now we state the following

Theorem . *Let $f \in \tilde{S}_0(K)$. Then the function F given by the formula:*

$$F(z) = \begin{cases} f(z) & \text{for } |z| \le 1 \\ f(\frac{1}{\bar z}) + \frac{|z|^2 - 1}{\bar z} \cdot f'(\frac{1}{\bar z})[1 - \omega(\frac{1}{\bar z})]^{-1} & \text{for } |z| > 1 \end{cases} \tag{3.2}$$

belongs to S_K, with $F(\infty) = \infty$, F is a K–q.c. extension of the function f.

Proof. Without loss of generality we can suppose that f is a holomorphic function in $\bar E$.

Let

$$g(z) = \overline{f(z) + \tau(z)p(z)f'(z)}\,,$$

where $p(z) = [1 - \omega(z)]^{-1}$ and $\omega(z)$ satisfies the inequality (3.1).

We get

$$|g'_{\bar z}| = |pf'|\left|\frac{1 + \tau'_z p}{p} + \frac{\tau}{z}\left(\frac{zp'}{p} + \frac{zf''}{f'}\right)\right|, \tag{3.3}$$

$$|g'_z| = |\tau'_{\bar z} pf'|\,. \tag{3.4}$$

The K–$q.c.$ condition : $|g'_{\bar z}| \le k|g'_z|$ with respect to (3.3) and (3.4) can be rewritten in the form:

$$\left|\frac{1 - \tau'_z p}{\tau'_{\bar z} p} + \frac{\tau}{z\,\tau'_{\bar z}}\left(\frac{zp'}{p} + \frac{zf''}{f'}\right)\right| \le k$$

By the relation $p(z) = [1 - \omega(z)]^{-1}$ we get

$$\left|\frac{1 + \tau'_z - \omega}{\tau'_{\bar z}} + \frac{\tau}{z\,\tau'_{\bar z}}\left(\frac{z\omega'}{1 - \omega} + \frac{zf''}{f'}\right)\right| \le k\,. \tag{3.5}$$

Let us remark that $\tau = \tau(z) = \dfrac{1 - |z|^2}{\bar z}$ satisfies the inequality (3.5), $\tau = 0$ for $|z| = 1$, $\tau'_{\bar z} = -\dfrac{1}{\bar z^2} \ne 0$ and this function we substitute in the definition of the function g.

The only point $z \in E$ such that $\tau = \infty$ is $z = 0$. Since $\tau'_{\bar z} \ne 0$ for $z \ne 0$, it follows that g is a local homeomorphism for $z \in E \setminus \{0\}$. By the equalities:

$$\frac{\partial}{\partial z}\left(\frac{1}{g(z)}\right)\bigg|_{z=0} = 1 - \omega(0) \ne 0 \quad \text{and} \quad \frac{\partial}{\partial \bar z}\left(\frac{1}{g(z)}\right)\bigg|_{z=0} = 0$$

we see that $F(z)$ given by formula (3.2) satisfies the thesis. Till now we have assumed that $f \in \tilde{S}_0(K)$ is a homeomorphism in $\bar E$.

Now we assume that f is a function of the class $\tilde{S}_0(K)$. In view of the lemma this theorem can be applied to the functions $f_r(z)$ and $\omega_r(z)$. Then we get

$$F_r(z) = \begin{cases} f_r(z) & \text{for } |z| \le 1 \\ f_r(\frac{1}{\bar z}) + \frac{|z|^2 - 1}{\bar z} \cdot f'_r(\frac{1}{\bar z}) \cdot (1 - \omega_r(z))^{-1} & \text{for } |z| > 1 \end{cases}$$

Since $f_r(z) \underset{r\to 1}{\longrightarrow} f$, $\omega_r(z) \underset{r\to 1}{\longrightarrow} \omega$ and the convergence is uniform on compact set E and since the class S_K is compact it follows that $F \in S_K$ which achives the proof.

Remark . It is easy to show that in the class $\tilde{S}_0$ the following structural formula holds:

$$f'(z) = \frac{1-\omega(z)}{1-\omega(0)} \exp\left[\int_0^z \frac{\varphi(t)}{t}\,dt\right] ,$$

where φ satisfies the condition:

$$\left|\omega(z)|z|^2 - (1-|z|^2)\varphi(z)\right| \le 1 , \quad \omega \in \Omega_0 \quad \text{and} \quad \varphi(0) = 0 .$$

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STRESZCZENIE

W pracy w oparciu o warunek dostateczny jednolistności funkcji holomorficznych klasy H podano charakterystykę pewnych podklas funkcji jednolistnych klasy S_0 i *K–q.c.* przedłużenie.

SUMMARY

In connection with a sufficient condition of univalence for functions holomorphic in the unit disk established by Lewandowski, cf. (1.2), some subclasses of the corresponding class S_0 of univalent functions and their quasiconformal extensions are ivestigated.