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On some Univalence Criteria

O pewnych kryteriach jednolistności

Let H denote a class of functions regular in the unit disk $K = \{z : |z| < 1\}$ and such that $f(0) = 0$ and $f'(z)f(z)/z \neq 0$ in K, and let G consist of all non-univalent functions belonging to the class H.

Let N and R denote the set of natural and real numbers, respectively. Consider an arbitrary non-negative functional Φ defined on the class H and let $a(\Phi) = \inf\{\Phi(f) : f \in G\}$. If it turns out that $\inf\{\Phi(f) : f \in H\} < a(\Phi)$, then we get the following simple univalence criterion : the condition $\Phi(f) < a(\Phi)$ for $f \in H$ implies the univalence of the function f in K.

In this paper we will give an estimate of constants $a(\Phi)$, where Φ has the form

(1) $$\Phi_{\alpha,\beta}(f) = \sup\{|\log[f'(z)(f(z)/z)^{\alpha+i\beta-1}]| : z \in K\}$$

or

(2) $$\Psi_{\alpha,\beta}(f) = \sup\{\log|f'(z)(f(z)/z)^{\alpha+i\beta-1}| - \log|f'(u)(f(u)/u)^{\alpha+i\beta-1}| : z, u \in K\}$$
for $\alpha, \beta \in R$.

Put $a(\Phi_{\alpha,\beta}) = a_{\alpha,\beta}$, $a(\Psi_{\alpha,\beta}) = b_{\alpha,\beta}$.

The number $a_{1,0}$ is known as the Robertson constant. In [3] J. Krzyż proved that $\pi/2 \leq a_{1,0} < 1.940$. The number $\exp b_{1,0}$ is called the John constant. It is known that $b_{1,0} \geq \pi/2$ (see, e.g. [1]). In the paper [2] J. Gevirtz proved that $\exp b_{1,0} < 7.189$. In the present paper we shall establish :

Theorem 1. *For the constants $a_{\alpha,\beta}$ we have the following estimates :*

(3) $$a_{\alpha,\beta} \geq \begin{cases} \alpha\pi + \pi/2 & \text{for } -1/2 < \alpha < 0,\ \beta \in R, \\ \pi/2 & \text{for } \alpha \geq 0,\ \beta \in R, \end{cases}$$

(4) $$a_{\alpha,\beta} \leq a_{\frac{\alpha}{n},\frac{\beta}{n}} \leq a_{0,0} < 1.765 \qquad \text{for } \alpha, \beta \in R \text{ and } n \in N.$$

Theorem 2. *For the constants* $b_{\alpha,\beta}$ *we have the following estimates :*

(5) $$b_{\alpha,\beta} \le b_{\frac{\alpha}{n},\frac{\beta}{n}} \le b_{0,0} \qquad \text{for } \alpha,\beta \in R,\ n \in N,$$

(6) $$\pi/2 \le b_{\alpha,\beta} \le b_{0,0} < 0.71\pi\ .$$

To prove inequality (3) we shall use

Lemma . [1]. *Let* $\alpha,\beta \in R$. *If* $f \in H$ *satisfies the condition*

(7) $$\int_{\theta_1}^{\theta_2} \mathrm{Re}[z\, f''(z)/f'(z) + 1 + (\alpha + i\beta - 1)z\, f'(z)/f(z)]\,d\theta > -\pi$$

for $0 \le \theta_2 - \theta_1 \le 2\pi,\ 0 < r < 1$, *where* $z = re^{i\theta}$, *then* f *is univalent in* K.

Proof of Theorem 1. Assume that $f \in H$ and $\Phi_{\alpha,\beta}(f) < t$. Then the function $z \to w(z) = t^{-1}\log[f'(z)(f(z)/z)^{\alpha+i\beta-1}]$ is regular in K and $|w(z)| < 1$ for $z \in K$. Since

$$z\, f''(z)/f'(z) + 1 + (\alpha + i\beta - 1)z\, f'(z)/f(z) = tz\, w'(z) + \alpha + i\beta$$

and

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}[tr\, e^{i\theta}\, w'(re^{i\theta}) + \alpha + i\beta]\,d\theta > \begin{cases} -2t + 2\alpha\pi & \text{for } \alpha < 0,\ \beta \in R, \\ -2t & \text{for } \alpha \ge 0,\ \beta \in R, \end{cases}$$

for $0 \le \theta_2 - \theta_1 \le 2\pi,\ r \in (0,1)$, the lemma implies (3).

Let now f be an arbitrary function of the class G and $f_n(z) = \left(f(z^n)\right)^{1/n}$. Obviously, $f_n \in G$ and

$$\begin{aligned}
&\Phi_{\alpha,\beta}(f_n) = \Phi_{\frac{\alpha}{n},\frac{\beta}{n}}(f) = \\
&= \sup\{|\log[(z\, f'(z)/f(z))^{1-1/n}[f'(z)(f(z)/z)^{\alpha+i\beta-1}]^{1/n}]| : z \in K\} = \\
&= \sup\{(1-1/n)|\log[z\, f'(z)/f(z)]| + (1/n)|\log[f'(z)(f(z)/z)^{\alpha+i\beta-1}]| : z \in K\} \le \\
&\le (1-1/n)\,\Phi_{0,0}(f) + (1/n)\,\Phi_{\alpha,\beta}(f).
\end{aligned}$$

Hence, for any natural number n and any function $f \in G$ we have $a_{\alpha,\beta} \le \Phi_{\alpha,\beta}(f_n) = \Phi_{\frac{\alpha}{n},\frac{\beta}{n}}(f)$, that is, $a_{\alpha,\beta} \le a_{\frac{\alpha}{n},\frac{\beta}{n}}$ and $a_{\alpha,\beta} \le (1-1/n)\Phi_{0,0}(f) + (1/n)\Phi_{\alpha,\beta}(f)$.

Since $f \in G$, there is $0 < r < 1$ such that $f_r(z) \overset{\mathrm{df}}{=} f(rz) \in G$. Thus $\Phi_{\alpha,\beta}(f_r) < \infty$. Passing to the limit as $n \to \infty$ in the preceding inequality we obtain $a_{\alpha,\beta} \le \Phi_{0,0}(f_r)$, and finally $a_{\alpha,\beta} \le \Phi_{0,0}(f)$.

To complete the proof of inequality (4) we have to show that $a_{0,0} < 1.765$. To this aim consider the following function

$$f(z) = z\ \exp\int_0^z (e^{tu} - 1)u^{-1}\,du \quad , \text{ where } t = 1.765\ .$$

Putting $\theta = \arcsin(\pi/(2t))$ we get

$$\mathrm{Re}[e^{i\theta} f'(e^{i\theta})/f(e^{i\theta})] = 0$$

and

$$\operatorname{Arg} f(e^{i\theta}) = \theta + \int_0^1 (1/x)\sin(\pi x/2)\exp\left[(x/2)\sqrt{4t^2 - x^2}\right]dx >$$

$$> \theta + \int_0^1 \left(\frac{\pi}{2} - \frac{\pi^3 x^2}{8\cdot 3!} + \frac{\pi^5 x^4}{32\cdot 5!} - \frac{\pi^7 x^6}{128\cdot 7!}\right)\exp\left((x/2)\sqrt{4t^2 - x^2}\right)dx\ .$$

Simple calculation shows that Arg $f(e^{i\theta}) > \pi$. But f has real coefficients, hence $f \in G$. Thus $a_{0,0} < \Phi_{0,0}(f) = t$.

Corollary 1. *For the Robertson constant* $a_{1,0}$ *we have* $a_{1,0} < 1.765$. *This improves the result obtained by* J. Krzyż *in paper* [3] ($a_{1,0} < 1.940$) .

Proof of Theorem 2. To prove inequality (5) we proceed as in the proof of inequalities $a_{\alpha,\beta} \le a_{\frac{\alpha}{n},\frac{\beta}{n}} \le a_{0,0}$. From (5) and from the known estimate $b_{1,0} \ge \pi/2$ (see, e.g. [1] p.34) follows $b_{\frac{1}{n},0} \ge \pi/2$.

Let us now show that $b_{0,0} < 0.71\pi$. To this end consider the function

$$f(z) = z\ \exp\int_0^z \left[(1-iu)^{it}/(1+iu)^{it} - 1\right]u^{-1}\,du \quad , \text{ where } t = 0.71\ .$$

Putting $\theta = \arcsin[\tanh(\pi/(2t))]$ we obtain

$$\operatorname{Re}\left[e^{-i\theta}f'(e^{i\theta})/f(e^{i\theta})\right] = 0$$

and

$$\operatorname{Arg}\left[e^{-i\theta}f(e^{i\theta})\right] = \operatorname{Im}\log\left[e^{-i\theta}f(e^{i\theta})/f(1)\right] = \operatorname{Im}\int_1^{e^{i\theta}} \left[(1-iu)^{it}/(1+iu)^{it}-1\right]u^{-1}\,du\ ,$$

since f has real coefficients.

Substituting $u = e^{ix}$ we get

$$\operatorname{Arg} f(e^{i\theta}) = e^{\pi t/2}\int_0^\theta \cos\left(\frac{t}{2}\log\frac{1+\sin x}{1-\sin x}\right)dx =$$

$$= 2e^{\pi t/2}\int_0^{\pi/(2t)} \left[(\cos ty)e^{-y}/(1+e^{-2y})\right]dy =$$

$$= 2e^{\pi t/2}\sum_{n=1}^{\infty}(-1)^{n+1} \times \int_0^{\pi/(2t)} e^{(1-2n)y}\cos ty\,dy =$$

$$= 2e^{\pi t/2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(te^{(1-2n)\pi/(2t)} + 2n - 1\right) \times \left((2n-1)^2 + t^2\right)^{-1} =$$

$$= 2e^{\pi t/2}\sum_{n=1}^{\infty}(-1)^{n+1}te^{(1-2n)\pi/(2t)}/\left((2n-1)^2 + t^2\right) + \pi e^{\pi t}/(1+e^{\pi t})\ .$$

Hence

$$\operatorname{Arg} f(e^{i\theta}) > 2te^{\pi t/2}\left[e^{-\pi/(2t)}/(1+t^2) - e^{-3\pi/(2t)}/(9+t^2)\right] + \pi e^{\pi t}/(1+e^{\pi t}) > \pi\ .$$

This means that $f \in G$, whence $b_{0,0} < \Psi_{0,0}(f) = \iota$, and the proof of Theorem 2 is complete.

Corollary 2. *For the John constant we have* $\exp b_{1,0} < 9.305$. *This result is weaker than the bound obtained by* J. Gevirtz *in* [2] (*namely* $\exp b_{1,0} < 7.189$).

REFERENCES

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STRESZCZENIE

Niech H oznacza klasę funkcji regularnych w kole jednostkowym $K = \{z : |z| < 1\}$ takich, że $f(0) = 0$, $f'(z)f(z)/z \neq 0$ w K, i niech G będzie zbiorem wszystkich funkcji niejednolistnych z klasy H.

Tematem pracy są oszacowania stałych $a(\Phi) = \inf\{\Phi(f) : f \in G\}$, dla których funkcjonał Φ jest postaci :

$$\text{(1)} \qquad \sup\{|\log[f'(z)(f(z)/z)^{\alpha+i\beta-1}]| : z \in K\},$$

lub

$$\text{(2)} \qquad \sup\{\log|f'(z)(f(z)/z)^{\alpha+i\beta-1}| - \log|f'(u)(f(u)/u)^{\alpha+i\beta-1}| : z, u \in K\}.$$

SUMMARY

Let H be the class of functions regular in the unit disk $K = \{z : |z| < 1\}$ such that $f(0) = 0$, $f'(z)f(z)/z \neq 0$ in K and let $G \subset H$ be its subclass consisting of all non-univalent functions. In this paper the constants $a(\Phi) = \inf\{\Phi(f) : f \in G\}$, where the functional Φ has the form (1), or (2), are estimated.